On the image of Fourier-Jacobi expansion

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Borcherds product

On the symmetric domain of type IV, Borcherds has constructed automorphic forms by infinite product in his paper *Automorphic forms on $O_{s+2,2}(\mathbb{R})$ and infinite products* in 1995.

In his paper, he set the discrete group $O_{s+2,2}(\mathbb{Z})$. And he gave an open problem: *Extend the methods of this paper to level greater than 1*.

To show the automorphic property is not so hard.
To show the convergence is very hard !!

In Borcherds paper, he did very complicated calculation on asymptotic behaviors of the Fourier coefficients of modular forms to investigate the analytic continuation of the infinite product.

Our problem

Problem

Find an easy way to show the convergence of given series that satisfies a kind of automorphic property, and apply it to Borcherds product.

The symmetric domain of type IV is a domain defined from an indefinite quadratic space of signature $(s+2,2)$.

$s=0 \Rightarrow \mathbb{H}\times\mathbb{H}$: very special case

$s=1 \Rightarrow \mathbb{H}_2$: Siegel upper half space of degree 2

The simplest case

The simplest case : $s = 1$
(Siegel modular forms of degree 2)

$$\Gamma := \mathrm{Sp}_2(\mathbb{Z})$$

$\mathbb{M}_k$: space of Siegel modular forms of weight k
$\mathbb{J}_{k,m}$: space of Jacobi forms of weight k index m

Fourier-Jacobi expansion

$$\mathbb{M}_k \ni F(\tau, z, \omega) = \sum_{m=0}^{\infty} \varphi_m(\tau, z) \exp(2\pi\sqrt{-1}m\omega)$$

$$\mathbb{M}_k \ni F \mapsto \{\varphi_m\} \in \prod_{m=0}^{\infty} \mathbb{J}_{k,m}$$

This is not surjective. We determine the image of this map.

Main theorem

The element $S := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ induces $F(\tau, z, \omega) = (-1)^k F(\omega, z, \tau)$.

Theorem. (The image of Fourier-Jacobi expansion)

These two maps are surjective, hence bijective.

$$\mathrm{FJ} \ : \ \mathbb{M}_k \ni F \mapsto \{\varphi_m\}_{m=0}^{\infty} \in \left(\prod_{m=0}^{\infty} \mathbb{J}_{k,m} \right)^{\mathrm{sym}}$$

$$\mathrm{FJ}^{\mathrm{c}} \ : \ \mathbb{M}_k^{\mathrm{c}} \ni F \mapsto \{\varphi_m\}_{m=1}^{\infty} \in \left(\prod_{m=1}^{\infty} \mathbb{J}_{k,m}^{\mathrm{c}} \right)^{\mathrm{sym}}$$

If $\{\varphi_m\}$ is in the image, the series $\sum_{m=0}^{\infty} \varphi_m(\tau, z) \exp\left(2\pi\sqrt{-1} m\omega\right)$ converges.

Application

Application

- convergence of Maass lift
- convergence of Borcherds product

Reference

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Siegel upper half space of degree 2

We denote Siegel upper half space of degree 2 by

$$\mathbb{H}_2 := \left\{ Z = {}^tZ = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathrm{M}_2(\mathbb{C}) \;\middle|\; \operatorname{Im} Z > 0 \right\}.$$

The symplectic group

$$G := \mathrm{Sp}_2(\mathbb{R}) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{M}_4(\mathbb{R}) \;\middle|\; {}^tMJM = J := \begin{pmatrix} O_2 & -E_2 \\ E_2 & O_2 \end{pmatrix} \right\}$$

acts on $\mathbb{H}_2$ transitively by

$$\mathbb{H}_2 \ni Z \longmapsto M\langle Z \rangle := (AZ+B)(CZ+D)^{-1} \in \mathbb{H}_2.$$

Siegel modular form of degree 2

For a holomorphic function $F:\mathbb{H}_2 \to \mathbb{C}$ and $k \in \mathbb{Z}$, define

$$(F|_k M)(Z) := \det(CZ+D)^{-k} F(M\langle Z \rangle).$$

In today's talk, we set $\Gamma := \mathrm{Sp}_2(\mathbb{Z}) := \mathrm{Sp}_2(\mathbb{R}) \cap \mathrm{M}_4(\mathbb{Z})$.

Definition. (Siegel modular form of degree 2)

We say F is a Siegel modular form of weight k if a holomorphic function F on $\mathbb{H}_2$ satisfies the condition $F = F|_k M$ for any $M \in \Gamma$. We denote the space of all Siegel modular forms of weight k by $\mathbb{M}_k$.

For simplicity, we denote $F(Z) = F(\tau, z, \omega)$.

Koecher principle

Let $F \in \mathbb{M}_k$. F has a Fourier expansion

$$F(Z) = \sum_{n,l,m} a(n,l,m)\, q^n \zeta^l p^m,$$

where $q^n := \mathbf{e}(n\tau) := \exp\left(2\pi\sqrt{-1}n\tau\right)$, $\zeta^l := \mathbf{e}(lz)$ and $p^m := \mathbf{e}(m\omega)$.

Proposition. (Koecher principle)

$a(n,l,m) = 0$ if $4mn - l^2 < 0$ or $m < 0$.

Definition. (Siegel cusp form)

we say $F \in \mathbb{M}_k$ is a cusp form of weight k if F satisfies the condition $a(n,l,m) = 0$ unless $4mn - l^2 > 0$. We denote the space of all cusp forms of weight k by $\mathbb{M}_k^{\mathrm{c}}$.

Jacobi group

The element $T := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ induces $F(\tau, z, \omega) \mapsto F(\tau, z, \omega + 1)$.

Define $G^J := \{M \in G \mid M^{-1}TM = T\}$. ($G = \mathrm{Sp}_2(\mathbb{R})$)

If $F : \mathbb{H}_2 \to \mathbb{C}$ has a period 1 with respect to ω, then $F|_k M$ also has a period 1 for any $M \in G^J$.

Let $m \in \mathbb{Z}$ and $\varphi(\tau, z)$ be a holomorphic function on $\mathbb{H} \times \mathbb{C}$. The image of $\varphi(\tau, z)p^m$ by $M \in G^J$ is a product of a holomorphic function on $\mathbb{H} \times \mathbb{C}$ and p^m. ($p^m = \exp\left(2\pi\sqrt{-1}m\omega\right)$)

Proposition. (Action of Jacobi group)

For each $m \in \mathbb{Z}$, the group G^J acts on the set of all holomorphic functions on $\mathbb{H} \times \mathbb{C}$.

Jacobi group invariant function

Define $\Gamma^J := G^J \cap \Gamma$. $\quad$ ($\Gamma = \mathrm{Sp}_2(\mathbb{Z})$)

Let $m \in \mathbb{Z}$ and $\varphi(\tau, z)$ be a holomorphic function on $\mathbb{H} \times \mathbb{C}$. We assume that $\varphi(\tau, z)p^m$ is Γ^J-invariant.

Namely, $\varphi(\tau, z)$ satisfies the following two equations:

$$\varphi(\tau, z) = (c\tau + d)^{-k}\mathbf{e}\left(\frac{-mcz^2}{c\tau + d}\right)\varphi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right)$$

$$\varphi(\tau, z) = \mathbf{e}\left(m\left(x^2\tau + 2xz\right)\right)\varphi\left(\tau, z + x\tau + y\right)$$

$$\left(\ \mathbf{e}(x) = \exp\left(2\pi\sqrt{-1}x\right)\ \right)$$

for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and for any $x, y \in \mathbb{Z}$.

(c.f. the book by Eichler and Zagier)

No negative index Jacobi forms

According to the book of Eichler and Zagier, we have the following propositions.

Proposition. (Jacobi form with negative index is zero)

If $m < 0$, above φ should be the zero function.

Proposition. (Jacobi form with zero index is constant)

If $m = 0$, above φ does not depend on z. Namely, it is a $\mathrm{SL}_2(\mathbb{Z})$-invariant holomorphic function on $\mathbb{H}$.

Hence we may assume $m \geq 0$.

Fourier expansion of Jacobi forms

Above φ has a Fourier expansion

$$\varphi(\tau, z) = \sum_{n,l} c(n,l)\, q^n \zeta^l. \qquad (q^n := \mathbf{e}(n\tau),\ \zeta^l := \mathbf{e}(lz))$$

If $m = 0$, $c(n,l) = 0$ for $l \neq 0$.

If $m > 0$, $c(n,l)$ depends only on $4mn - l^2$ and $l \pmod{2m}$.

Especially, if $m = 1$, $c(n,l)$ depends only on $4mn - l^2$ and sometimes we denote $c(4mn - l^2) = c(n,l)$.

Jacobi form

Definition (Jacobi form Jacobi cusp form weak Jacobi form w.h. Jacobi form)

We say above φ is a Jacobi cusp form weak Jacobi form w.h. Jacobi form of weight k and index m if $c(n,l)=0$ except when $n>0$ and $4mn-l^2>0$ $n\geq 0$ $n \geq -^{\exists}N$.

We denote the space of all Jacobi cusp form weak Jacobi form w.h. Jacobi form of weight k and index m by $\mathbb{J}^{\mathrm{c}}_{k,m}$ $\mathbb{J}^{\mathrm{w}}_{k,m}$ $\mathbb{J}^{\mathrm{wh}}_{k,m}$.

Jacobi form	$n\geq 0$ and $4mn-l^2\geq 0$	$\mathbb{J}_{k,m}$
Jacobi cusp form	$n>0$ and $4mn-l^2>0$	$\mathbb{J}^{\mathrm{c}}_{k,m}$
weak Jacobi form	$n\geq 0$	$\mathbb{J}^{\mathrm{w}}_{k,m}$
w.h. Jacobi form	$n\geq -^{\exists}N$	$\mathbb{J}^{\mathrm{wh}}_{k,m}$

(w.h. $\cdots$ weakly holomorphic)

Property of Jacobi forms (1)

For $\varphi \in \mathbb{J}^{\mathrm{wh}}_{k,m}$, a positive valued function

$$G_\varphi(\tau, z) := |\varphi(\tau, z)| \exp\left(\frac{-2\pi m (\operatorname{Im} z)^2}{\operatorname{Im} \tau}\right) (\operatorname{Im} \tau)^{\frac{k}{2}}$$

is Γ^J-invariant, namely $G_\varphi|_{0,0} M = G_\varphi$ for any $M \in \Gamma^J$.

Proposition. (Upper bound of a Jacobi cusp form)

If $\varphi \in \mathbb{J}^{\mathrm{c}}_{k,m}$, G_φ has a maximum value.

Proposition. (Upper bound of Fourier coefficients)

For $\varphi \in \mathbb{J}^{\mathrm{c}}_{k,m}$, there exists a constant K_φ such that $|c(n,l)| \le K_\varphi \left(4mn - l^2\right)^{\frac{k}{2}}$.

$$\left(\quad \varphi(\tau, z) = \sum_{n,l} c(n,l)\, q^n \zeta^l, \qquad (q^n := \mathbf{e}(n\tau),\ \zeta^l := \mathbf{e}(lz)) \quad \right)$$

Property of Jacobi forms (2)

If $m = 0$, φ is a $\mathrm{SL}_2(\mathbb{Z})$-invariant holomorphic function on $\mathbb{H}$. Hence $\mathbb{J}^{\mathrm{w}}_{k,0} = \mathbb{J}_{k,0} = \mathbb{A}_k$ and $\mathbb{J}^{\mathrm{c}}_{k,0} = \{0\}$, where $\mathbb{A}_k$ is a space of all elliptic modular forms of weight k w.r.t. $\mathrm{SL}_2(\mathbb{Z})$.

$\mathbb{J}^{\mathrm{w}}_{*,*} := \bigoplus_{k,m\in\mathbb{Z}} \mathbb{J}^{\mathrm{w}}_{k,m}$ is a graded ring of $\mathbb{A}_* := \bigoplus_{k,m\in\mathbb{Z}} \mathbb{A}_k$.

Proposition. (Structure theorem of weak Jacobi forms)

$\mathbb{J}^{\mathrm{w}}_{*,*}$ is generated by $\varphi_{0,1}$, $\varphi_{-2,1}$ and $\varphi_{-1,2}$ on $\mathbb{A}_*$.

(We remark that $\mathbb{J}_{*,*}$ is not finitely generated on $\mathbb{A}_*$).

Fourier-Jacobi expansion

The Fourier Jacobi expansion is a p-expansion of $F \in \mathbb{M}_k$ or $\mathbb{M}_k^{\mathrm{c}}$:

$$F(Z) = \sum_{m=0}^{\infty} \varphi_m(\tau, z) p^m.$$

For $F \in \mathbb{M}_k$, we have $\varphi_m \in \mathbb{J}_{k,m}$.
For $F \in \mathbb{M}_k^{\mathrm{c}}$, we have $\varphi_0 = 0$ and $\varphi_m \in \mathbb{J}_{k,m}^{\mathrm{c}}$.

Definition. (Fourier-Jacobi expansion)

The Fourier Jacobi expansion is a map from $\mathbb{M}_k$ or $\mathbb{M}_k^{\mathrm{c}}$ to the infinite direct product space of Jacobi forms:

$$\mathrm{FJ} \ : \ \mathbb{M}_k \ni F \mapsto \{\varphi_m\}_{m=0}^{\infty} \in \prod_{m=0}^{\infty} \mathbb{J}_{k,m}$$

$$\mathrm{FJ}^{\mathrm{c}} \ : \ \mathbb{M}_k^{\mathrm{c}} \ni F \mapsto \{\varphi_m\}_{m=1}^{\infty} \in \prod_{m=1}^{\infty} \mathbb{J}_{k,m}^{\mathrm{c}}$$

But these two maps are not surjective !!

The symmetry

The element $S := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ induces $F(\tau, z, \omega) \mapsto (-1)^k F(\omega, z, \tau)$.
$S \in \Gamma$ gives the information about the image of the Fourier-Jacobi expansion. Namely, on the Fourier expansion

$$\mathbb{M}_k \ni F(Z) = \sum_{n,l,m} a(n,l,m)\; q^n \zeta^l p^m,$$

we have $a(n,l,m) = (-1)^k a(m,l,n)$.

Proposition. (Generators of the symplectic group)

The group $G = \mathrm{Sp}_2(\mathbb{R})$ is generated by G^J and S.
The group $\Gamma = \mathrm{Sp}_2(\mathbb{Z})$ is generated by Γ^J and S.

The image of the Fourier-Jacobi expansion

Let

$$\left(\prod_{m=0}^{\infty} \mathbb{J}_{k,m}\right)^{\mathrm{sym}} := \left\{ \{\varphi_m\} \in \prod_{m=0}^{\infty} \mathbb{J}_{k,m} \;\middle|\; c_m(n,l) = (-1)^k c_n(m,l) \right\}$$

$$\left(\prod_{m=1}^{\infty} \mathbb{J}^{\mathrm{c}}_{k,m}\right)^{\mathrm{sym}} := \left\{ \{\varphi_m\} \in \prod_{m=1}^{\infty} \mathbb{J}^{\mathrm{c}}_{k,m} \;\middle|\; c_m(n,l) = (-1)^k c_n(m,l) \right\}$$

where we denote $\varphi_m(\tau, z) = \sum_{n,l} c_m(n,l)\, q^n \zeta^l$.

Theorem. (The image of Fourier-Jacobi expansion)

These two maps are surjective, hence bijective.

$$\mathrm{FJ} \;:\; \mathbb{M}_k \ni F \mapsto \{\varphi_m\}_{m=0}^{\infty} \in \left(\prod_{m=0}^{\infty} \mathbb{J}_{k,m}\right)^{\mathrm{sym}}$$

$$\mathrm{FJ}^{\mathrm{c}} \;:\; \mathbb{M}^{\mathrm{c}}_k \ni F \mapsto \{\varphi_m\}_{m=1}^{\infty} \in \left(\prod_{m=1}^{\infty} \mathbb{J}^{\mathrm{c}}_{k,m}\right)^{\mathrm{sym}}$$

Flow of our proof

Theorem. (The image of Fourier-Jacobi expansion)

These two maps are surjective, hence bijective.

$$\mathrm{FJ} \ : \ \mathbb{M}_k \ni F \mapsto \{\varphi_m\}_{m=0}^{\infty} \in \left(\prod_{m=0}^{\infty} \mathbb{J}_{k,m} \right)^{\mathrm{sym}}$$

$$\mathrm{FJ}^{\mathrm{c}} \ : \ \mathbb{M}_k^{\mathrm{c}} \ni F \mapsto \{\varphi_m\}_{m=1}^{\infty} \in \left(\prod_{m=1}^{\infty} \mathbb{J}_{k,m}^{\mathrm{c}} \right)^{\mathrm{sym}}$$

Does $\sum_{m=0}^{\infty} \varphi_m(\tau, z) p^m$ converge absolutely and locally uniformly on $\mathbb{H}$?

- Step 1. Estimation of Jacobi cusp forms on ‘rational’ points
- Step 2. Convergence at ‘rational’ points
- Step 3. Complete the proof of FJ^{c}.
- Step 4. Complete the proof of FJ.

Step 1. (1)

First, we suppose $\{\varphi_m\}_{m=1}^{\infty} \in \left(\prod_{m=1}^{\infty} \mathbb{J}_{k,m}^{c} \right)^{\mathrm{sym}}$.

Take $x, y \in \mathbb{Q}$ with same denominator D. Then

$$f_m(\tau) := \mathbf{e}(mx^2\tau)\varphi_m(\tau, x\tau + y)$$

is an elliptic cusp form of weight k with respect to the main congruent subgroup of level D^2. The Fourier expansion of f_m is

$$f_m(\tau) = \sum_{\nu > 0} a_m(\nu)\, q^{\nu} \qquad \Bigg(a_m(\nu) = \sum_{\substack{n,l \in \mathbb{Z} \\ mx^2 + lx + n = \nu}} c_m(n, l) \Bigg).$$

We remark that the number of the pair (n, l) satisfying $mx^2 + lx + n = \nu$ is less than $4\sqrt{m\nu} + 1$. These (n, l) satisfies the condition $4mn - l^2 \leq 4m\nu$.

Step 1. (2)

Because the space of the above elliptic cusp forms is finite dimensional, there exist L and C such that

$$|f_m(\tau)| \leq C \left(\sum_{\nu \leq L} |a_m(\nu)| \right) (\operatorname{Im} \tau)^{-\frac{k}{2}} .$$

Hence we have

$$G_m(\tau, x\tau + y) \leq C \left\{ \sum_{\nu \leq L} \Bigg(\sum_{\substack{n,l \in \mathbb{Z} \\ mx^2 + lx + n = \nu}} |c_m(n,l)| \Bigg) \right\},$$

where G_m means G_{φ_m}.

Step 2. (1)

Here we denote the Fourier coefficient $c\begin{pmatrix} n & l/2 \\ l/2 & m \end{pmatrix} = c_m(n,l)$.

For any $A \in \mathrm{SL}_2(\mathbb{Z})$, $\begin{pmatrix} {}^tA & O_2 \\ O_2 & A^{-1} \end{pmatrix} \in \Gamma$ induces $c(T) = c({}^tATA)$.

For $x = \frac{\alpha}{\beta}$, take $A = \left(\begin{smallmatrix} * & \beta \\ * & \alpha \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Then we have

$$c\begin{pmatrix} n & l/2 \\ l/2 & m \end{pmatrix} = c\left({}^tA \begin{pmatrix} n & l/2 \\ l/2 & m \end{pmatrix} A\right) = c\begin{pmatrix} * & * \\ * & n\beta^2+l\alpha\beta+m\alpha^2 \end{pmatrix}.$$

Because $n\beta^2 + l\alpha\beta + m\alpha^2 = (mx^2 + lx + n)\beta^2 \leq (mx^2 + lx + n)D^2$, we regard that m in $|c_m(n,l)|$ at Step 1 is sufficiently small.

Step 2. (2)

Namely, from

$$G_m(\tau, x\tau + y) \leq C \left\{ \sum_{\nu \leq L} \Bigg(\sum_{\substack{n,l \in \mathbb{Z} \\ mx^2 + lx + n = \nu}} |c_m(n,l)| \Bigg) \right\},$$

there exists a constant K such that

$$G_m(\tau, x\tau + y) \leq CK \left\{ \sum_{\nu \leq L} \Bigg(\sum_{\substack{n,l \in \mathbb{Z} \\ mx^2 + lx + n = \nu}} (4mn - l^2)^{\frac{k}{2}} \Bigg) \right\},$$

namely, there exists a constant C' (depends on D) such that

$$G_m(\tau, x\tau + y) \leq C' m^{\frac{k+1}{2}}$$

Step 3.

Proposition. (Structure theorem of weak Jacobi forms)

$\mathbb{J}^{\mathrm{w}}_{*,*}$ is generated by $\varphi_{0,1}$, $\varphi_{-2,1}$ and $\varphi_{-1,2}$ on $\mathbb{A}_*$.

For any $R > 1$ and $(\tau_0, z_0) \in \mathbb{H} \times \mathbb{C}$, there exist its neighbourhood U and $x, y \in \mathbb{Q}$ such that $G_m(\tau, z) \leq R^m C' m^{\frac{k+1}{2}}$ for any $(\tau, z) \in U$. Namely,

$$|\varphi(\tau, z)| \leq R^m C' m^{\frac{k+1}{2}} \exp\left(\frac{2\pi m (\operatorname{Im} z)^2}{\operatorname{Im} \tau}\right) (\operatorname{Im} \tau)^{-\frac{k}{2}}.$$

Hence we know the series $\displaystyle\sum_{m=1}^{\infty} \varphi_m(\tau, z) p^m$ is holomorphic on $\mathbb{H}_2$.

$\left(\text{Under the assumption } \{\varphi_m\}_{m=1}^{\infty} \in \left(\prod_{m=1}^{\infty} \mathbb{J}^{\mathrm{c}}_{k,m}\right)^{\mathrm{sym}}.\right)$

Step 4.

Let $\Delta_{10} \in \mathbb{M}_{10}^{c}$ be a unique Siegel cusp form of weight 10. We remark that $\Delta_{10}(\tau, 0, \omega) = 0$.

Proposition. (Freitag)

If $F \in \mathbb{M}_k$ $(k \in 2\mathbb{Z})$ satisfies $F(\tau, 0, \omega) = 0$, then $F/\Delta_{10} \in \mathbb{M}_{k-10}$.

Now suppose $k \in 2\mathbb{Z}$, $\{\varphi_m\}_{m=0}^{\infty} \in \left(\prod_{m=0}^{\infty} \mathbb{J}_{k,m}\right)^{\mathrm{sym}}$ and regard $F := \sum_{m=0}^{\infty} \varphi_m(\tau, z) p^m$ as a power series of p.

Then each coefficients of p^m on $F\Delta_{10}$ is a Jacobi cusp form, hence by Step 3, $F\Delta_{10}$ is holomorphic.

Therefore, by above proposition, F is holomorphic.

For odd k, we have a similar proof. (use Δ_{35})

Convergence of Maass lift

Theorem. (Maass Lift)

For any $\varphi \in \mathbb{J}^{\mathrm{c}}_{k,1}$, there exists $F \in \mathbb{M}^{\mathrm{c}}_k$ such that $\mathrm{FJ}^{\mathrm{c}}_1(F) = \varphi$.

The Hecke operator $T_-(m)$ induces a map from $\mathbb{J}^{\mathrm{c}}_{k,1}$ to $\mathbb{J}^{\mathrm{c}}_{k,m}$.

$$\left(\varphi | T_-(m)\right)(\tau, z) := \sum_{ad=m} \sum_{b=0}^{d-1} a^k \varphi\left(\frac{a\tau + b}{d}, az\right)$$

The series

$$\begin{aligned} F(Z) &:= \sum_{m=1}^{\infty} \frac{1}{m} \left(\varphi | T_-(m)\right)(\tau, z) p^m \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{l} \left(\sum_{a|(n,l,m)} a^{k-1} c\left(\frac{mn}{a^2}, \frac{l}{a}\right)\right) q^n \zeta^l p^m \end{aligned}$$

converges by our main theorem.

Lift of a weakly holomorphic Jacobi form

Now let $\varphi(\tau, z) = \sum_{n,l} c(4mn - l^2)\, q^n \zeta^l \in \mathbb{J}_{0,1}^{\mathrm{wh}}$.

Calculate the Maass lift of φ, although it does not converge:

$$\begin{aligned}
&\sum_{m=1}^{\infty} \sum_{n=-N}^{\infty} \sum_{l} \left(\sum_{a|(n,l,m)} a^{-1} c\left(\frac{4mn - l^2}{a^2} \right) \right) q^n \zeta^l p^m \\
&= \sum_{m=1}^{\infty} \sum_{n=-N}^{\infty} \sum_{l} c(4mn - l^2) \left(\sum_{a=1}^{\infty} a^{-1} \left(q^n \zeta^l p^m \right)^a \right) \\
&= \sum_{m=1}^{\infty} \sum_{n=-N}^{\infty} \sum_{l} c(4mn - l^2) \left(-\log\left(1 - q^n \zeta^l p^m\right) \right).
\end{aligned}$$

Borcherds product

Hence, **formally**,

$$\prod_{m=1}^{\infty}\prod_{n=-N}^{\infty}\prod_{l}\left(1-q^n\zeta^l p^m\right)^{c(4mn-l^2)}$$

is a Γ^J-invariant function of weight 0. By slight modification, Borcherds constructs a Γ-invariant function of weight $c(0)/2$:

$$p^a\zeta^b q^c\prod_{(m,n,l)>0}\left(1-q^n\zeta^l p^m\right)^{c(4mn-l^2)},$$

where $a=\frac{1}{2}\sum_{l>0}l^2c(-l^2)$, $b=-\frac{1}{2}\sum_{l>0}lc(-l^2)$, $c=\frac{1}{24}\sum_{l\in\mathbb{Z}}c(-l^2)$ and $(m,n,l)>0$ means $m>0$ or $m=0,n>0$ or $m=n=0,l>0$.

Convergence

This infinite product (Borcherds product) is a Γ-invariant function and has a symmetry.

However, generally, this infinite product (Borcherds product) does not converge. Borcherds has investigated its analytic continuation. He has determined all zero and poles of this product and shown it to be a meromorphic modular form on $\mathbb{H}_2$.

By our main theorem, if we show each coefficient of the p-expansion of this infinite product, it should be a holomorphic modular form.

In fact, using this method, we can find out the Borcherds product Δ_{10}, Δ_{35} and so on.

With levels

Consider the case

$$\Gamma = \Gamma_0(N) := \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{Z}) \;\middle|\; C \equiv O_2 \pmod{N} \right\}$$

$\Gamma_0(N)$ is generated by $\Gamma_0(N)^J$ and S.

We can show the first part of our main theorem in a similar way.

Theorem. (The image of Fourier-Jacobi expansion)

The map FJ is surjective, hence bijective.

$$\mathrm{FJ} \;:\; \mathbb{M}_k \ni F \mapsto \{\varphi_m\}_{m=0}^{\infty} \in \left(\prod_{m=0}^{\infty} \mathbb{J}_{k,m} \right)^{\mathrm{sym}}$$

This gives a partial answer of Borcherds open problem: *Extend the methods of this paper to level greater than 1.*

Automorphic forms on O(2,s+2)

Generalize our main theorem to automorphic forms on the symmetric domain of type IV:

$$\mathcal{H} := G/K \quad (K = \mathrm{O}(2) \times \mathrm{O}(s+2) : \text{max. cpt.}) \quad \curvearrowleft \quad \Gamma$$

Obstacles

- Step 2 : Can we make m so small ?
  (Is the Fourier group sufficiently large?)
- Step 3 : Is the space of Jacobi forms always finitely generated?
- Step 4 : Does the good cusp forms like Δ_{10} always exist?

Thank you

Thank you for your kind attention.