On the structure theorem for modular forms … Igusa's result and beyond

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A conference in memory of Jun-ichi Igusa

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Professor Jun-ichi Igusa for me (1)

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Keywords:
Primitive forms, Period integral by Kyoji Saito
Siegelsch Modulfunktionen by E. Freitag
Jacobi forms by D. Zagier, M. Eichler, ...

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He is Igusa's high school junior!
And he was a professor on physics in my university.

Siegel upper half space

We denote Siegel upper half space of degree 2 by

$$\mathbb{H}_2 := \left\{ Z = {}^tZ = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathrm{M}_2(\mathbb{C}) \;\middle|\; \operatorname{Im} Z > 0 \right\}.$$

The symplectic group

$$\mathrm{Sp}_2(\mathbb{R}) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{M}_4(\mathbb{R}) \;\middle|\; {}^tMJM = J := \begin{pmatrix} O_2 & -E_2 \\ E_2 & O_2 \end{pmatrix} \right\}$$

acts on $\mathbb{H}_2$ transitively by

$$\mathbb{H}_2 \ni Z \longmapsto M\langle Z\rangle := (AZ+B)(CZ+D)^{-1} \in \mathbb{H}_2.$$

For a holomorphic function $F : \mathbb{H}_2 \to \mathbb{C}$ and $k \in \mathbb{Z}$, define

$$(F|_k M)(Z) := \det(CZ+D)^{-k} F(M\langle Z\rangle).$$

Siegel modular forms

Let $\mathrm{Sp}_2(\mathbb{Z}) := \mathrm{Sp}_2(\mathbb{R}) \cap \mathrm{M}_4(\mathbb{Z})$.

Definition.

We say a holomorphic function $F : \mathbb{H}_2 \to \mathbb{C}$ is a Siegel modular form of weight k if F satisfies the condition $F|_k M = F$ for any $M \in \mathrm{Sp}_2(\mathbb{Z})$.

$\mathbb{M}_k$: $\mathbb{C}$-vector space of all Siegel modular forms of weight k.

From general theory, we can show:

- If $k < 0$, $\mathbb{M}_k = \{0\}$.
- If $k = 0$, $\mathbb{M}_0 = \mathbb{C}$.
- If $k > 0$, $\dim_{\mathbb{C}} \mathbb{M}_k$ is finite.

Graded ring of Siegel modular forms

Then,

$$\mathbb{M}_* := \bigoplus_{k \in \mathbb{Z}} \mathbb{M}_k$$

is a graded ring.

Question.

To determine the structure of the graded ring $\mathbb{M}_*$

From general theory, we know there are 4 algebraically independent generators, however, to determine the explicit structure of $\mathbb{M}_*$ is not easy.

Igusa's theorem

Theorem. (Igusa 1962)

- $\mathbb{M}_*$ is generated by 5 forms of weight $4, 6, 10, 12, 35$.
- The first 4 generators are algebraically independent.
- The square of the last generator is in the ring generated by the first 4 generators. (He showed the explicit relations.)

$$\sum_{k=0}^{\infty} (\dim \mathbb{M}_k)\, x^k = \frac{1 + x^{35}}{(1 - x^4)(1 - x^6)(1 - x^{10})(1 - x^{12})}$$

This is the first result on the determination of the structure of modular forms of several variables.

Graded ring of modular forms

My interest

To determine the structure of the graded ring of modular forms of several variables.

This is closely related to:

- Dimension formula
- Construction of modular forms
  - Theta functions
  - Differential operators
  - Eisenstein series
- Fourier coefficients of modular forms
  - L-functions
  - Hecke theory
- Number theory
- Algebraic Geometry

General theory

Here we consider modular forms of n variables.

$\mathbb{M}_k$: $\mathbb{C}$-vector space of all modular forms of weight $k \in \mathbb{Z}$.

Under suitable condirtions, we can show:

- If $k < 0$, $\mathbb{M}_k = \{0\}$.
- If $k = 0$, $\mathbb{M}_0 = \mathbb{C}$.
- If $k > 0$, $\dim_{\mathbb{C}} \mathbb{M}_k$ is finite.

Then,

$$\mathbb{M}_* := \bigoplus_{k \in \mathbb{Z}} \mathbb{M}_k$$

is a graded ring.

There are $n + 1$ algebraically independent generators of $\mathbb{M}_*$.

Difficulties

What are difficulties to determine the structure of $\mathbb{M}_*$?

- How to determine the exact dimension of $\mathbb{M}_k$?
- How to construct generators? (Especially lower weight case)

In case of $\mathrm{Sp}_2(\mathbb{Z})$, Igusa resolved these difficulties for the first time.

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To find new way is much easier than to find the first way.

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To find new way is much easier than to find the first way.

Now we know many proofs of Igusa's theorem:

- Igusa (1962) : Origin
- Freitag (1965) : Zeros of the theta products
- A. (2000) : Jacobi forms
- van der Geer (2008) : Diagonal restriction

Structure theorem

To find new way is much easier than to find the first way.

Igusa's work stimurated the determination of the ring of modular forms of many kinds.

- Hilbert modular forms on real quadratic field
  - Gundlach (1963) : $\mathbb{Q}(\sqrt{5})$
  - Hammond, Hirzebruch, ... : $\mathbb{Q}(\sqrt{D})$
  - A. (2001) : $\mathbb{Q}(\sqrt{5})$ mixed weight
- Siegel modular forms of degree 2
  - Satoh (1986), Ibukiyama (2001), ... : vector valued
  - Igusa, Ibukiyama, Hayashida, Gunji, A., ... : with levels
  - Ibukiyama and Onodera (1997), Ibukiyama, Poor, Yuen (2013), ... : paramodular forms
- Siegel modular forms of degree 3
  - Tsuyumine (1986)
- Hermitian modular forms of degree 2
  - Freitag (1967) : $\mathbb{Q}\sqrt{-1}$
  - Dern(1996) : $\mathbb{Q}\sqrt{-3}$
- Modular forms on $\mathrm{O}(2, n+2)$
  - Krieg, Freitag, Salvati Manni, ...

Igusa's theorem

$\mathbb{M}_*$: Graded ring of all Siegel modular forms w.r.t. $\mathrm{Sp}_2(\mathbb{Z})$

Theorem. (Igusa 1962)

- $\mathbb{M}_*$ is generated by 5 forms of weight $4, 6, 10, 12, 35$.
- The first 4 generators are algebraically independent.
- The square of the last generator is in the ring generated by the first 4 generators. (He showed the explicit relations.)

$$\mathbb{M}_* = R \oplus \chi_{35} R, \qquad R = \mathbb{C}[E_4, E_6, \chi_{10}, \chi_{12}]$$

What are difficulties to determine the structure of $\mathbb{M}_*$?

- How to determine the exact dimension of $\mathbb{M}_k$?
- How to construct generators $E_4, E_6, \chi_{10}, \chi_{12}$ and χ_{35}?

How to determine the exact dimension?

How to determine the exact dimension of $\mathbb{M}_k$?

- algebraic geometry
- trace formula
- another way

$$\sum_{k=0}^{\infty} (\dim \mathbb{M}_k)\, x^k = \frac{1 + x^{35}}{(1-x^4)(1-x^6)(1-x^{10})(1-x^{12})}$$

How to construct generators?

How to construct generators $E_4, E_6, \chi_{10}, \chi_{12}$ and χ_{35}?

- Eisenstein Series $(E_4, E_6, \chi_{10}, \chi_{12})$
- Theta constants $(E_4, E_6, \chi_{10}, \chi_{12}, \chi_{35})$ (by Igusa)
- Saito-Kurokawa lift, Maass lift $(E_4, E_6, \chi_{10}, \chi_{12})$
- Rankin-Cohen-Ibukiyama differential operator (χ_{35})
- Borcherds product (χ_{10}, χ_{35})

$$E_4 = \mathrm{ML}(e_{4,1}), \qquad E_6 = \mathrm{ML}(e_{6,1})$$
$$\chi_{10} = \mathrm{ML}(\varphi_{10,1}), \qquad \chi_{12} = \mathrm{ML}(\varphi_{12,1})$$

$$\chi_{35} = \det \begin{pmatrix} 4E_4 & 6E_6 & 10\chi_{10} & 12\chi_{12} \\ \frac{\partial}{\partial\tau}E_4 & \frac{\partial}{\partial\tau}E_6 & \frac{\partial}{\partial\tau}\chi_{10} & \frac{\partial}{\partial\tau}\chi_{12} \\ \frac{\partial}{\partial z}E_4 & \frac{\partial}{\partial z}E_6 & \frac{\partial}{\partial z}\chi_{10} & \frac{\partial}{\partial z}\chi_{12} \\ \frac{\partial}{\partial\omega}E_4 & \frac{\partial}{\partial\omega}E_6 & \frac{\partial}{\partial\omega}\chi_{10} & \frac{\partial}{\partial\omega}\chi_{12} \end{pmatrix}$$

Fourier expansion

Because $\begin{pmatrix} 1 & 0 & s & t \\ 0 & 1 & t & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{Z})$ $(s, t, u \in \mathbb{Z})$, $F \in \mathbb{M}_k$ satisfies

$$F\begin{pmatrix} \tau + s & z + t \\ z + t & \omega + u \end{pmatrix} = F\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix}.$$

Hence $F \in \mathbb{M}_k$ has a **Fourier-expansion**

$$\mathbb{M}_k \ni F(Z) = \sum_{n,l,m \in \mathbb{Z}} a(n,l,m) q^n \zeta^l p^m.$$

$$\left(\quad q^n := \mathbf{e}(n\tau) := \exp(2\pi\sqrt{-1}n\tau), \quad \zeta^l := \mathbf{e}(lz), \quad p^m := \mathbf{e}(m\omega) \quad \right)$$

Proposition. (Koecher principle)

If $m < 0$ or if $4nm - l^2 < 0$, then $a(n,l,m) = 0$.

Fourier-Jacobi expansion

On **Fourier-Jacobi expansion**

$$\mathbb{M}_k \ni F(Z) = \sum_{m \in \mathbb{Z}} \varphi_m(\tau, z) p^m,$$

each $\varphi_m(\tau, z)p^m$ is invarinat under the (weight k) action of

$$\mathrm{Sp}_2^{\mathrm{J}}(\mathbb{Z}) := \left\{ M \in \mathrm{Sp}_2(\mathbb{Z}) \;\middle|\; M \begin{pmatrix} 1&0&0&0\\ 0&1&0&1\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix} M^{-1} = \begin{pmatrix} 1&0&0&0\\ 0&1&0&1\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix} \right\}.$$

Definition.

We say a holomorphic function $\varphi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ is a **Jacobi form** of weight k and index m if $\varphi(\tau, z)p^m$ is invariant under the weight k action of $\mathrm{Sp}_2^{\mathrm{J}}(\mathbb{Z})$ and satisfies the Koecher principle.

$\mathbb{J}_{k,m}$: $\mathbb{C}$-vector space of all Jacobi forms of weight k and index m.

Jacobi forms

Jacobi forms

We say a holomorphic function $\varphi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ is a **Jacobi form** of weight k and index m if φ satisfies the following three conditions:

- $\varphi(\tau, z) = (c\tau + d)^{-k} \mathbf{e}\left(\frac{-mcz^2}{c\tau + d}\right) \varphi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right)$ for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.
- $\varphi(\tau, z) = \mathbf{e}\left(m\left(x^2\tau + 2xz\right)\right) \varphi\left(\tau, z + x\tau + y\right)$ for any $x, y \in \mathbb{Z}$.
- On the Fourier expansion $\varphi(\tau, z) = \sum_{n,l \in \mathbb{Z}} c(n,l) q^n \zeta^l$, $c(n,l) = 0$ if $n < 0$ or if $4nm - l^2 < 0$.

  $$\left(q^n := \mathbf{e}(n\tau) := \exp\left(2\pi\sqrt{-1}n\tau\right),\ \zeta^l := \mathbf{e}(lz)\right)$$

Here we assume $k, m \in \mathbb{Z}$.

M. Eichler and D. Zagier, *The theory of Jacobi forms*, Birkhäuser, 1985.

Jacobi forms and weak Jacobi forms

Definition.

We say a holomorphic function $\varphi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ is a **weak Jacobi form** of weight k and index m if $\varphi(\tau, z)p^m$ is invariant under the weight k action of $\mathrm{Sp}_2^{\mathrm{J}}(\mathbb{Z})$ and $c(n, l) = 0$ $(n < 0)$, where

$$\varphi(\tau, z) = \sum_{n,l \in \mathbb{Z}} c(n, l) q^m \zeta^l.$$

$\mathbb{J}^{\mathrm{w}}_{k,m}$: space of all weak Jacobi forms of weight k and index m.

We can show:

- If $m < 0$, $\mathbb{J}^{\mathrm{w}}_{k,m} = \mathbb{J}_{k,m} = \{0\}$.
- If $m = 0$, $\mathbb{J}^{\mathrm{w}}_{k,0} = \mathbb{J}_{k,0} = \mathrm{M}_k$ (space of elliptic modular forms).
- If $m > 0$, $\mathbb{J}^{\mathrm{w}}_{k,m} \supset \mathbb{J}_{k,m}$ and $\dim_{\mathbb{C}} \mathbb{J}^{\mathrm{w}}_{k,m}$ is finite.

Structure of weak Jacobi forms

Here,

$$\mathbb{J}^{\mathrm{w}}_{*,*} := \bigoplus_{k,m\in\mathbb{Z}} \mathbb{J}^{\mathrm{w}}_{k,m} \quad \text{and} \quad \mathbb{J}_{*,*} := \bigoplus_{k,m\in\mathbb{Z}} \mathbb{J}_{k,m}$$

are bi-graded rings.

Theorem. (M. Eichler and D. Zagier (1985))

$\mathbb{J}_{*,*}$ is not finitely generated over $\mathbb{C}$, but

$$\mathbb{J}^{\mathrm{w}}_{*,*} = R \oplus \varphi_{-1,2} R, \quad R = \mathrm{M}_*[\varphi_{0,1}, \varphi_{-2,1}].$$

Remark. (well known)

The structure of the graded ring of elliptic modular forms is

$$\mathrm{M}_* := \bigoplus_{k\in\mathbb{Z}} \mathrm{M}_k = \mathbb{C}[e_4, e_6].$$

Siegel modular forms and Jacobi forms

Now we have

$$\mathbb{M}_k \ni F(Z) = \sum_{m=0}^{\infty} \varphi_m(\tau, z) p^m \quad \Longrightarrow \quad \varphi_m \in \mathbb{J}_{k,m} \subset \mathbb{J}^{\mathrm{w}}_{k,m}.$$

$(\mathrm{Sp}_2(\mathbb{Z})$ invariant $)$ $(\mathrm{Sp}_2^{\mathrm{J}}(\mathbb{Z})$ invariant $)$

Proposition.

$\mathrm{Sp}_2(\mathbb{Z})$ is generated by $\mathrm{Sp}_2^{\mathrm{J}}(\mathbb{Z})$ and $\begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{pmatrix}$.

$$\begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{pmatrix} \text{ induces } F\begin{pmatrix} \tau & z\\ z & \omega \end{pmatrix} = (-1)^k F\begin{pmatrix} \omega & z\\ z & \tau \end{pmatrix}.$$

Proof of Igusa's theorem

On Fourier(-Jacobi) expansion

$$\mathbb{M}_k \ni F(Z) = \sum_{m=0}^{\infty} \varphi_m(\tau, z) p^m$$
$$= \sum_{n,l,m \in \mathbb{Z}} a(n,l,m) q^n \zeta^l p^m$$

we have

$$a(n,l,m) = (-1)^k a(m,l,n).$$

Therefore, we have an injection

$$\mathbb{M}_k \ni F \mapsto (\varphi_m)_{m=0}^{\infty} \in \left(\prod_{m=0}^{\infty} \mathbb{J}_{k,m} \right)^{\mathrm{sym}}$$

and

$$\sum_{k=0}^{\infty} \left(\dim \left(\prod_{m=0}^{\infty} \mathbb{J}_{k,m} \right)^{\mathrm{sym}} \right) x^k = \frac{1 + x^{35}}{(1-x^4)(1-x^6)(1-x^{10})(1-x^{12})}.$$

Another proof

Key point of the proof

Reduce the number of variables, keep the discrete subgroup not so smaller.

This way (A. (2000))

Siegel modular forms	$\to$	Jacobi forms
	$\to$	Elliptic modular forms

Siegel paramodular forms with level ≤ 4 : Ibukiyama, Poor, Yuen

Another way (van der Geer (2008))

Siegel modular forms	$\to$	Modular forms on $\mathbb{H} \times \mathbb{H}$
	$\to$	Elliptic modular forms

Siegel modular forms with level ≤ 4 : A., Ibukiyama

Similar way is avairable for Hilbert modular forms and Hermitian modular forms.

There is a simple unified structure...

Looking many results of the structure of the graded ring of modular forms, we find there is a simple unified structure.

For example:

Theorem. (Ibukiyama and A. (2005))

The graded ring of Siegel modular forms of degree 2 with level $N \leqq 4$ has a very simple unified structure.
(For $N = 3, 4$, we take a character.)

- There are 5 generators.
- The first 4 generators are algebraically independent.
- The last generator is obtained from the first 4 generators by using Rankin-Cohen-Ibukiyama differential operator.

Sigel modular forms of degree 2 with levels

$$\sum_{k=0}^{\infty} (\dim \mathbb{M}_k(\mathrm{Sp}_2(\mathbb{Z})))\, x^k = \frac{1+x^{35}}{(1-x^4)(1-x^6)(1-x^{10})(1-x^{12})}$$

$$\sum_{k=0}^{\infty} (\dim \mathbb{M}_k(\Gamma_0(2)))\, x^k = \frac{1+x^{19}}{(1-x^2)(1-x^4)(1-x^4)(1-x^6)}$$

$$\sum_{k=0}^{\infty} (\dim \mathbb{M}_k(\Gamma_0'(3)))\, x^k = \frac{1+x^{14}}{(1-x)(1-x^3)(1-x^3)(1-x^4)}$$

$$\sum_{k=0}^{\infty} (\dim \mathbb{M}_k(\Gamma_0'(4)))\, x^k = \frac{1+x^{11}}{(1-x)(1-x^2)(1-x^2)(1-x^3)}$$

Why unified??

Siegel paramodular forms of degree 2 with levels

Another example:

Theorem. (A. (2016))

Let Γ_N be a suitable subgroup of Siegel paramodular group of level $N = 2, 3, 4$. The graded ring of modular forms of Γ_N has a very simple unified structure.
(We take a character.)

- There are 6 generators of weights $4, 6, \frac{12}{N} - 2, \frac{12}{N}, \frac{24}{N} - 1, 12$.
- The first 4 generators are algebraically independent.
- The first 5 generators are obtained by a kind of Maass lift.
- The last generator is obtained from the first 5 generators by using Rankin-Cohen-Ibukiyama differential operator.

Siegel paramodular forms of degree 2 with levels

$$\sum_{k=0}^{\infty} (\dim \mathbb{M}_k(\Gamma_2))\, x^k = \frac{\left(1+x^{11}\right)\left(1+x^{12}\right)}{(1-x^4)(1-x^4)(1-x^6)(1-x^6)}$$

$$(4+4+6+6+3=11+12)$$

$$\sum_{k=0}^{\infty} (\dim \mathbb{M}_k(\Gamma_3))\, x^k = \frac{\left(1+x^{7}\right)\left(1+x^{12}\right)}{(1-x^2)(1-x^4)(1-x^4)(1-x^6)}$$

$$(2+4+4+6+3=7+12)$$

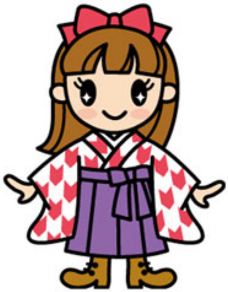
$$\sum_{k=0}^{\infty} (\dim \mathbb{M}_k(\Gamma_4))\, x^k = \frac{\left(1+x^{5}\right)\left(1+x^{12}\right)}{(1-x)(1-x^3)(1-x^4)(1-x^6)}$$

$$(1+3+4+6+3=5+12)$$

Why unified??

Thank you

Thank you for your kind attention.



BOCCHAN and MADONNACHAN are the mascots of Tokyo University of Science.