

Odd perfect numbers have a prime factor exceeding 10^8 *

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Abstract

In this article, it is shown that every odd perfect number is divisible by a prime greater than 10^8 . The method of the proof is based on the paper by Hagis and Cohen. By improving on the original algorithms, we could shorten computing time. Some property of cyclotomic polynomials is used in our algorithms.

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1 Introduction

A positive integer n is said to be *perfect* if $\sigma(n) = 2n$, where $\sigma(n)$ denotes the sum of positive divisors of n . As of January 2006, forty three even perfect numbers are known, however, it is still open whether or not odd one does exist. Many necessary conditions for their existence have been found. For example, Euler showed that the prime decomposition of an odd perfect number n must be the form of

$$n = p^e p_1^{2e_1} \cdots p_t^{2e_t}, \quad p \equiv e \equiv 1 \pmod{4}.$$

This p is called the *special prime* of n . Brent, Cohen and te Riele [1] showed that $n > 10^{300}$. Hagis [6] and Chein [2] independently showed that n must have at least 8 distinct prime factors, and Hare [8] showed that n must have totally at least 47 prime factors.

In this article, we focus our attention on the largest prime factor of an odd perfect number. Suppose that n is an odd perfect number and P_n is the largest prime factor of n . In 1944, Kanold [11] showed that $P_n > 60^{\S}$. This bound was extended by Hagis and McDaniel [4] (resp. [5]) to $P_n > 10^4$ (resp. $P_n > 10^5$), by Hagis and Cohen [7] to $P_n > 10^6$, by Jenkins [9], [10] to $P_n > 10^7$. Jenkins reported that he needed about 25800 hours for computing time. The aim of this article is to show the following result.

Theorem 1.1 *If n is an odd perfect number and P_n is the largest prime factor, then $P_n > 10^8$.*

In order to show this theorem, it was necessary to improve on the original algorithms. In §3.1, we discuss the improvement of the algorithm, and give the data of CPU time in §A.2.

2 Cyclotomic numbers

Let d be a positive integer. The d -th cyclotomic polynomial $\Phi_d(X)$ is defined by

$$\Phi_d(X) = \prod_{\substack{j \in \{1, \dots, d\} \\ (j, d) = 1}} (X - e^{2\pi\sqrt{-1}j/d}).$$

It is well-known that $\Phi_d(X) \in \mathbb{Z}[X]$. We often deal with the case that d is equal to a prime r , in which

$$\Phi_r(X) = X^{r-1} + X^{r-2} + \cdots + X + 1.$$

For a positive integer a , the integer $\Phi_d(a)$ is called a *cyclotomic number*. For a prime p , it is easily verified that

$$\sigma(p^e) = \prod_{\substack{d \mid (e+1) \\ d \neq 1}} \Phi_d(p),$$

hence cyclotomic numbers are important for a study of odd perfect numbers. Assume that n is an odd perfect number, and its prime decomposition is $p_1^{e_1} \cdots p_k^{e_k}$. From $2n = \sigma(n)$, it immediately follows that

$$2 \prod_{i=1}^k p_i^{e_i} = \prod_{i=1}^k \prod_{\substack{d \mid (e_i+1) \\ d \neq 1}} \Phi_d(p_i). \tag{2.1}$$

In this section, we recall the properties of cyclotomic numbers.

In 19-th century, cyclotomic numbers were studied by Sylvester, Kronecker et al. The following proposition is a summary of their results (cf. [14], [15]).

[§]It is considered that he did not use a computer. The second author showed that the largest prime factor of an odd *harmonic number* must be greater than 100 (cf. [3]), without a computer. This is an extension of Kanold's result.

Proposition 2.1 *Let p be a prime, and a, d integers. Suppose that $a \geq 2, d \geq 3$. Then we have the following.*

- (1) $p \mid \Phi_d(a) \Rightarrow p \mid d$ or $p \equiv 1 \pmod{d}$.
- (2) $p \mid \Phi_d(a), p \mid d \Rightarrow p^2 \nmid \Phi_d(a)$.
- (3) *Every cyclotomic number $\Phi_d(a)$ has at least one prime factor p such that $p \equiv 1 \pmod{d}$, except the case that $\Phi_6(2) = 3$.*

Our improved algorithms are based on the following proposition.

Proposition 2.2 *Let p, q, r be primes, and suppose that $q \mid \Phi_r(p), q \equiv 1 \pmod{r}$. Let g be a generator of the cyclic group $(\mathbb{Z}/q^m\mathbb{Z})^\times$, and put $t = (q-1)/r$. Then $q^m \mid \Phi_r(p)$ if and only if p belongs to the subgroup $\langle g^{tq^{m-1}} \rangle$ of $(\mathbb{Z}/q^m\mathbb{Z})^\times$.*

proof. Note that t is an integer since $q \equiv 1 \pmod{r}$. Assume that $p \equiv 1 \pmod{q}$. Then it follows that $\Phi_r(p) \equiv r \pmod{q}$, a contradiction. Hence $p \not\equiv 1 \pmod{q}$ and

$$\begin{aligned} q^m \mid \Phi_r(p) &\iff q^m \mid (p-1)\Phi_r(p) \\ &\iff p^r \equiv 1 \pmod{q^m} \\ &\iff \text{the order of } p \in (\mathbb{Z}/q^m\mathbb{Z})^\times \text{ is } r \\ &\iff p \in \langle g^{tq^{m-1}} \rangle, \end{aligned}$$

as required. □

3 Outline of the proof

In this section, we give an outline of the proof of Theorem 1.1. We discuss details of the proof in the appendix. In this article, p, q, r will always denote primes.

3.1 Acceptable values

Assume that n is an odd perfect number and the largest prime factor of n is less than 10^8 . Then the left-hand side of (2.1) has only prime factors less than 10^8 , hence so does the right-hand side. Since $2n \equiv 2 \pmod{4}$, the cyclotomic numbers in (2.1) are not divided by 4.

Definition For an odd prime p and a prime r , we say that $\Phi_r(p)$ is *acceptable* if and only if the following two conditions hold.

- (1) $\Phi_r(p)$ has no prime factor greater than 10^8 .
- (2) $4 \nmid \Phi_r(p)$.

Clearly, the cyclotomic numbers in (2.1) must be acceptable. Our first aim (but the hardest task) is to find all acceptable values.

Lemma 3.1 *Suppose that $r \geq 7, 3 \leq p < 10^8$ and the cyclotomic number $\Phi_r(p)$ is acceptable. Then $\Phi_r(p)$ is one of 671 numbers listed in §A.5.*

The complete proof of this lemma is in §A.1. From Proposition 2.1 (3), the cyclotomic number $\Phi_r(p)$ has a prime factor q such that $q \geq 2r + 1$, hence if $\Phi_r(p)$ is acceptable, then $r < 5 \cdot 10^7$. Therefore we need to investigate only finitely many numbers, however, it is hard to directly judge each number acceptable (or unacceptable) because of difficulty of prime factorization. The key point of the proof of Lemma 3.1 is the following fact. In this article, $p^e \parallel n$ means that $p^e \mid n$ and $p^{e+1} \nmid n$.

Lemma 3.2 *If $p, q < 10^8$, $6679 < r < 5 \cdot 10^7$, then $q^4 \nmid \Phi_r(p)$, and $q^3 \parallel \Phi_r(p)$ for at most one q for each $\Phi_r(p)$. In fact, $q^3 \nmid \Phi_r(p)$ except that*

$$28499^3 \parallel \Phi_{14249}(70081199), \quad 60647^3 \parallel \Phi_{30323}(6392117), \quad 63587^3 \parallel \Phi_{31793}(42326917).$$

We can check this fact using UBASIC program `cubeprg.ub` or PARI/GP program `cubeprg.gp`. Since checking this fact needs much time, it is important to improve the algorithm. For each pair (r, q) with $q \equiv 1 \pmod{r}$, we need to search p such that $q^2 \mid \Phi_r(p)$. The original algorithm moves p for whole range, however, Proposition 2.2 says that $q^2 \mid \Phi_r(p)$ is equivalent to $p \in \langle g^{q(q-1)/r} \rangle \subset (\mathbb{Z}/q^2\mathbb{Z})^\times$, hence, we need move p for only this range. This improvement shortens CPU time very much. For details, see §A.2.

3.2 Inadmissible small primes

Under Lemma 3.1, we show the following.

Lemma 3.3 *Suppose that an odd perfect number n has no prime factor greater than 10^8 . Then n has no prime factor which is in the following set X .*

$$X = \{3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 43, 61, 71, 113, 127, 131, 151, 197, 211, 239, 281, 337, \\ 379, 421, 449, 463, 491, 547, 617, 631, 659, 673, 743, 757, 827, 911, 953, 967, 1051, 1093\}.$$

In particular, n has no prime factor less than 41.

For the proof of this lemma, primes in X are considered in the order

$$1093, 151, 31, 127, 19, 11, 7, 23, 131, 37, 61, 13, 3, 5, 29, 43, 1051, 17, 71, 113, 197, 211, 239 \\ 281, 337, 379, 421, 449, 463, 491, 547, 617, 631, 659, 673, 743, 757, 827, 911, 953, 967.$$

For example, after proving that $1093 \nmid n$, we prove $151 \nmid n$ as follows. Assume that $151 \mid n$. Then the right-hand side of (2.1) is divided by a cyclotomic number $\Phi_r(151)$. Only $\Phi_3(151) = 3 \cdot 7 \cdot 1093$ is acceptable, however, it is contradictory to $1093 \nmid n$. Hence we have $151 \nmid n$. The complete proof is given in §A.3.

3.3 Restriction on exponents in the prime decomposition

Under Lemmas 3.1, 3.3, we show the following.

Lemma 3.4 *Suppose that an odd perfect number n has no prime factor greater than 10^8 , and the prime decomposition of n is $n = p_1^{e_1} \cdots p_r^{e_r}$. Then the product $\prod(e_i + 1)$ has no prime factor greater than 5.*

Assume that some $e_i + 1$ has a prime factor r greater than 5. Then the cyclotomic number $\Phi_r(p_i)$ is acceptable. Hence $\Phi_r(p_i)$ is one of the numbers listed in §A.5 by Lemma 3.1. From Lemma 3.3, it follows that $p_i \notin X$, and $\Phi_r(p_i)$ has no prime factor which is in X . There are 87 such cyclotomic numbers. It is sufficient to show that such numbers do not appear in the equation (2.1).

In this section, we check only three $\Phi_r(p_i)$ with $r \geq 11$, and check other numbers in §A.4.

$$\Phi_{13}(47) = 53 \cdot 2237 \cdot 14050609 \cdot 71265169, \\ \Phi_{13}(83) = 1249 \cdot 1396513 \cdot 1423319 \cdot 43580447, \\ \Phi_{11}(691) = 59951 \cdot 133717 \cdot 183041 \cdot 455489 \cdot 37187767.$$

(1) If $\Phi_{13}(47) \mid n$, then $14050609 \mid n$. Only $\Phi_2(14050609) = 2 \cdot 5 \cdot 7 \cdot 200723$ is acceptable, however, $5 \in X$, a contradiction.

- (2) If $\Phi_{13}(83) \mid n$, then $1396513 \mid n$. Only $\Phi_2(1396513) = 2 \cdot 7 \cdot 23 \cdot 4337$ is acceptable, however, $7 \in X$, a contradiction.
- (3) If $\Phi_{11}(691) \mid n$, then $133717 \mid n$. Only $\Phi_2(133717) = 2 \cdot 13 \cdot 37 \cdot 139$ is acceptable, however, $13 \in X$, a contradiction.

3.4 Four sets

In this section, we complete the proof of Theorem 1.1. Let $P = \{p \mid 41 \leq p < 10^8\}$. Using the program `ptestest.ub`, we have $\#P = 5761443$ and

$$P^* := \prod_{p \in P} \frac{p}{p-1} < 4.87934286481804236682.$$

We define subsets S, T, U, V of P as follows.

$$\begin{aligned} S &= \{p \in P \mid p \not\equiv 1 \pmod{3} \text{ and } p \not\equiv 1 \pmod{5}\}, \\ T &= \{p \in P \mid p \equiv 1 \pmod{15}\}, \\ U &= \{p \in P \mid p \equiv 1 \pmod{3}, p \not\equiv 1 \pmod{5} \text{ and } \Phi_5(p) \text{ has a prime factor greater than } 10^8\}, \\ V &= \{p \in P \mid p \not\equiv 1 \pmod{3}, p \equiv 1 \pmod{5} \text{ and } \Phi_3(p) \text{ has a prime factor greater than } 10^8\}. \end{aligned}$$

Note that these subsets are disjoint. Using the programs `stestest.ub`, `ttestest.ub`, `utestest.ub` and `vtestest.ub`, we have $\#S = 2160618$, $\#T = 719983$, $\#U = 2144188$, $\#V = 496701$ and

$$\begin{aligned} S^* &:= \prod_{p \in S} \frac{p}{p-1} > 1.82219345901032950583, \\ T^* &:= \prod_{p \in T} \frac{p}{p-1} > 1.19902263543776496408, \\ U^* &:= \prod_{p \in U} \frac{p}{p-1} > 1.43699138263382743310, \\ V^* &:= \prod_{p \in V} \frac{p}{p-1} > 1.03750936160818766647. \end{aligned}$$

The following proposition can be proven similarly to Hagis and Cohen [7].

Proposition 3.5 *Suppose that an odd perfect number n has no prime factor greater than 10^8 . Then the following hold.*

- (1) *The number n is divisible by at most two elements of S . If there is such an element s , then it is not the special prime of n , and $s \geq 47$.*
- (2) *The number n is divisible by at most one element of T . If there is such an element t , then it is the special prime of n , and $t \geq 61$.*
- (3) *The number n is divisible by at most one element of U . If there is such an element u , then it is the special prime of n , and $u \geq 73$.*
- (4) *The number n is not divisible by any element of V .*

proof. We show only (1) since it is slightly different from the original one. Suppose that $p \in S$ and $p \mid n$. Then p divides a cyclotomic number $\Phi_d(p_j)$ in the equality (2.1). Assume that p_j is not the special prime. Then d is odd. From Lemma 3.4, d is divisible by only 3 or 5, hence, $p \equiv 1 \pmod{3}$ or $p \equiv 1 \pmod{5}$ by Proposition 2.1. This is a contradiction to $p \in S$. Therefore p_j is the special prime, and p is not special. Since the smallest element of S is 47, it holds that $p \geq 47$. Since p is not special and divides only $\Phi_2(p_j)$, we have $p^2 \mid (p_j + 1)$. If there are three such p , then $p_j + 1 \geq 2 \cdot 47^2 \cdot 53^2 \cdot 59^2 > 10^8$, a contradiction to $p_j < 10^8$. \square

We define $\sigma_{-1}(n)$ by

$$\sigma_{-1}(n) := \sum_{d|n} d^{-1} = \frac{\sigma(n)}{n}.$$

The integer n is perfect if and only if $\sigma_{-1}(n) = 2$. The function σ_{-1} is multiplicative, that is, if $(a, b) = 1$, then $\sigma_{-1}(ab) = \sigma_{-1}(a)\sigma_{-1}(b)$. It is easily verified that if $p < q$, then $\sigma_{-1}(p^e) > \sigma_{-1}(q^f)$ for any positive integers e, f , and

$$\sigma_{-1}(p^e) < \sigma_{-1}(p^\infty) := \lim_{m \rightarrow \infty} \sigma_{-1}(p^m) = \frac{p}{p-1}.$$

Assume that n is an odd perfect number whose largest prime factor is less than 10^8 . From Proposition 3.5, it follows that

$$2 = \sigma_{-1}(n) < \prod_i \frac{p_i}{p_i - 1} \leq \frac{47}{46} \cdot \frac{53}{52} \cdot \frac{61}{60} \cdot \frac{P^*}{S^*T^*U^*V^*} < 1.5859314817,$$

a contradiction. The proof of Theorem 1.1 is completed.

This inequality is much stronger than is needed and that the theorem could be proved by only S and U . The referee of the paper [7] also pointed out the same fact. However, it is considered that the four sets are valuable, as Hagis and Cohen claimed.

A Appendix

A.1 Details on the search for acceptable values

In the rest of this article, p, q, r will always denote primes less than 10^8 .

Lemma A.1 *If $10^2 < p < 10^8$ and $6679 < r < 5 \cdot 10^7$, then $\Phi_r(p)$ is unacceptable.*

proof. For a prime r , we define $Q(r)$ by

$$Q(r) = \prod_{p < 10^8, p \equiv 1 \pmod{r}} p.$$

First, we show

Claim 1. If $6679 < r < 5 \cdot 10^8$, then $(10^8)^2 \cdot r \cdot Q(r)^2 < 10^{2(r-1)}$.

Suppose that $r \geq 5 \cdot 10^4$. If $p \equiv 1 \pmod{r}$, then $p = 2kr + 1$ with $k < 10^3$, hence we have $10^8 \cdot \sqrt{r} \cdot Q(r) < (10^8)^{10^3+2} < 10^{10^4} < 10^{r-1}$, as required. The case that $6679 < r < 5 \cdot 10^4$ can be directly checked by the program `claim1.ub`.

Assume that $6679 < r < 5 \cdot 10^7$ and $\Phi_r(p)$ is acceptable. From Lemma 3.2, it follows that

$$\Phi_r(p) < (10^8)^2 \cdot r \cdot Q(r)^2 < 10^{2(r-1)}.$$

On the other hand, we have

$$\Phi_r(p) > p^{r-1} > 10^{2(r-1)},$$

a contradiction. □

Lemma A.2 *If $10^2 < p < 10^8$ and $4723 < r \leq 6679$, then $\Phi_r(p)$ is unacceptable.*

proof. By using the program `claim2.ub`, we can show

Claim 2. If $4723 < r \leq 6679$, then $10^8 r \cdot Q(r) < 10^{2(r-1)}$.

By the program `sqrprg.ub`, we can show

If $4723 < r \leq 6679$, then $q^3 \nmid \Phi_r(p)$, and $q^2 \parallel \Phi_r(p)$ for at most one q for each $\Phi_r(p)$.

Assume that $4723 < r \leq 6679$ and $\Phi_r(p)$ is acceptable. Then it follows that $10^{2(r-1)} < p^{r-1} < \Phi_r(p) < 10^8 \cdot r \cdot Q(r) < 10^{2(r-1)}$, a contradiction. \square

Lemma A.3 (1) *If $10^6 < p < 10^8$, $2707 < r \leq 4723$, then $\Phi_r(p)$ is unacceptable.*

(2) *If $10^7 < p < 10^8$, $2503 < r \leq 2707$, then $\Phi_r(p)$ is unacceptable.*

proof. By the program `sqrprg.ub`, we can show that if $3 \leq p, q < 10^8$, $2503 < r \leq 4723$, then $q^2 \mid \Phi_r(p)$ for at most one q for each $\Phi_r(p)$, and $q^3 \nmid \Phi_r(p)$ expect that

$$\begin{aligned} 10709^3 \parallel \Phi_{2677}(6619441), \quad 5939^3 \parallel \Phi_{2969}(41492783), \quad 6719^3 \parallel \Phi_{3359}(59698039), \\ 8147^3 \parallel \Phi_{4073}(41112823), \quad 8147^3 \parallel \Phi_{4073}(41728717). \end{aligned}$$

(1) By the program `claim3.ub`,

Claim 3. If $2707 < r \leq 4723$, then $(10^8)^2 \cdot r \cdot Q(r) < 10^{6(r-1)}$.

Hence if $2707 < r \leq 4723$ and $\Phi_r(p)$, then it follows that $10^{6(r-1)} < p^{r-1} < \Phi_r(p) < (10^8)^2 \cdot r \cdot Q(r) < 10^{6(r-1)}$, a contradiction.

(2) By the program `claim4.ub`,

Claim 4. If $2503 < r \leq 2707$, then $(10^8)^2 \cdot r \cdot Q(r) < 10^{7(r-1)}$.

Hence if $2503 < r \leq 2707$ and $\Phi_r(p)$, then it follows that $10^{7(r-1)} < p^{r-1} < \Phi_r(p) < (10^8)^2 \cdot r \cdot Q(r) < 10^{7(r-1)}$, a contradiction. \square

Suppose that $p < 10^2$. If $q^2 \mid \Phi_r(p)$, then $r \mid (q-1)$ and $p^r \equiv 1 \pmod{q^2}$, hence $p^{q-1} \equiv 1 \pmod{q^2}$. Conversely if $p^{q-1} \equiv 1 \pmod{q^2}$ and there is a prime factor r of $q-1$ such that $p^r \equiv 1 \pmod{q^2}$, then $q^2 \mid \Phi_r(p)$. According to the table of Montgomery [12], all solutions of $p^{q-1} \equiv 1 \pmod{q^2}$, $3 \leq p < 10^2$, $q < 10^8$ are given in the following table.

p	q	p	q
3	11, 1006003	43	5, 103
5	20771, 40487, 53471161	47	
7	5, 491531	53	3, 47, 59, 97
11	71	59	2777
13	863, 1747591	61	
17	3, 46021, 48947	67	7, 47, 268573
19	3, 7, 13, 43, 137, 63061489	71	3, 47, 331
23	13, 2481757, 13703077	73	3
29		79	7, 263, 3037, 1012573, 60312841
31	7, 79, 6451, 2806861	83	4871, 13691
37	3, 77867	89	3, 13
41	29, 1025273	97	7, 2914393

From the table, if $p < 10^2$, then $q^2 \nmid \Phi_r(p)$ except that

$$\begin{aligned} 11^2 \parallel \Phi_5(3), \quad 48947^2 \parallel \Phi_{24473}(17), \quad 47^2 \parallel \Phi_{23}(53), \quad 59^2 \parallel \Phi_{29}(53), \\ 7^2 \parallel \Phi_3(67), \quad 47^2 \parallel \Phi_{23}(71), \quad 7^2 \parallel \Phi_3(79), \quad 4871^2 \parallel \Phi_{487}(83). \end{aligned}$$

[¶]We also checked this by the program `cubeprg.gp`. The computer search showed that if $6679 < r < 5 \cdot 10^7$, $q < 10^8$, $a < 10^2$, then $q^2 \nmid \Phi_r(a)$ except that $48947^2 \parallel \Phi_{24473}(17)$, $401771^2 \parallel \Phi_{40177}(63)$.

For a prime p , we define $R(p)$ by

$$R(p) = \min\{a \in \mathbb{N} \mid r \geq a \Rightarrow 10^8 r \cdot Q(r) < p^{r-1}\}.$$

It immediately follows that $R(p) \leq 5 \cdot 10^4$. In fact, if $r \geq 5 \cdot 10^4$, then

$$10^8 r \cdot Q(r) < (10^8)^{10^3+2} < (3^{17})^{10^3+2} < 3^r \leq p^r.$$

By directly checking each $r < 5 \cdot 10^4$ (`rvalue.ub`), we have the following table.

p	$R(p)$	p	$R(p)$	p	$R(p)$
3	9650	29	5508	61	4952
5	7950	31	5508	67	4890
7	7238	37	5310	71	4878
11	6548	41	5262	73	4878
13	6318	43	5262	79	4818
17	5954	47	5108	83	4788
19	5882	53	5060	89	4734
23	5660	59	4988	97	4724

Lemma A.4 *If $3 \leq p < 10^2$ and $R(p) \leq r < 5 \cdot 10^7$, then $\Phi_r(p)$ is unacceptable.*

proof. Assume that $\Phi_r(p)$ is acceptable. Since $q^2 \nmid \Phi_r(p)$ for at most one q , it follows that $p^r < \Phi_r(p) < 10^8 r \cdot Q(r) < p^r$, a contradiction. \square

Now, we must check only r in the table below for each p .

p	r
$3 \cdots 10^2$	$R(p) - 1$
$10^5 \cdots 10^6$	4723
$10^6 \cdots 10^7$	2707
$10^7 \cdots 10^8$	2503

Using `accept.ub` for $7 \leq r \leq 4723$, `accept2.ub` for $3 \leq p < 10^2$, we can show Lemma 3.1.

A.2 Remarks on CPU time

We used the following two systems for the computations.

	machine	CPU	OS	software
A:	HP, AlpherServer GS320	Alpha21264, 731MHz	Tru64 UNIX V5.1	PARI/GP, GP2C
B:	DELL, Dimension8300	Pentium4, 3GHz	Windows XP Home Edition	UBASIC

The machine of A belongs to Computing and Communications Center, Kyushu University, one of B to the first author. We can use some CPU's simultaneously in the system A, and "CPU time" means the total time. We used the system A for checking Lemma 3.2, and the system B for the other computations.

Note that GP2C is available on only UNIX, and UBASIC on only Windows. The GP script `cubeprg.gp` in §A.6.1 is available on both UNIX and Windows. For the practical computation on system A, we used `cubeprg1.gp.c` which is a translated C program from `cubeprg1.gp` by GP2C.

In order to observe how faster our algorithm is than the original one, we compared CPU time in the following three conditions.

- I. system A and the C program `cubeprg1.gp.c`
- II. system B and the UBASIC program `cubeprg.ub`
- III. system B and the original program^{||}

Using each condition, we showd the following lemmas which are necessary for each bound $10^5, 10^6, 10^7, 10^8$.

Lemma A.5 *If $p, q < 10^5$ and $211 < r < 5 \cdot 10^4$, then $q^3 \nmid \Phi_r(p)$.*

Lemma A.6 (cf. [7]) *If $p, q < 10^6$ and $659 < r < 5 \cdot 10^5$, then $q^3 \nmid \Phi_r(p)$ except that*

$$3119^3 \parallel \Phi_{1559}(146917).$$

Lemma A.7 ([9],[10]) *If $p, q < 10^7$ and $2142 < r < 5 \cdot 10^6$, then $q^3 \nmid \Phi_r(p)$ except that*

$$10709^3 \parallel \Phi_{2677}(6619441), \quad 60647^3 \parallel \Phi_{30323}(6392117).$$

Lemma A.8 (Lemma 3.2) *If $p, q < 10^8$ and $6679 < r < 5 \cdot 10^7$, then $q^3 \nmid \Phi_r(p)$ except that*

$$28499^3 \parallel \Phi_{14249}(70081199), \quad 60647^3 \parallel \Phi_{30323}(6392117), \quad 63587^3 \parallel \Phi_{31793}(42326917).$$

The CPU time is as follows. The dashes (—) mean that the authors cannot test because of their hard tasks.

	I	II	III
Lemma A.5	about 3 minutes	about 30 seconds	about 5 minutes
Lemma A.6	about 3 hours	about 35 minutes	about 11 hours
Lemma A.7	about 274 hours	about 42 hours	—
Lemma A.8	about 26000 hours**	—	—

The list below gives CPU time of the other computations.

	program	CPU time
1.	<code>sqrprg.ub</code>	about an hour
2.	<code>rvalue.ub</code>	about 30 minutes
3.	<code>claim1.ub</code>	about a minute
4.	<code>claim2.ub</code>	less than a minute
5.	<code>claim3.ub</code>	less than a minute
6.	<code>claim4.ub</code>	less than a minute
7.	<code>accept.ub</code>	about 570 hours
8.	<code>accept2.ub</code>	about 200 minutes
9.	<code>ptester.ub</code>	about 30 minutes
10.	<code>stester.ub</code>	about 10 minutes
11.	<code>ttester.ub</code>	about 5 minutes
12.	<code>utester.ub</code>	about 2 hours
13.	<code>vtester.ub</code>	about an hour

Proposition 2.2 is also used in program 1,7,12 and 13, and shortens CPU time very much.

In order to renew the record to 10^9 , we need much CPU time or a better method. It is shown that if $\Phi_r(p)$ is acceptable, then $r \leq 47$. Can we eliminate the possibilities of $47 < r < 5 \cdot 10^7$ without hard computations? The authors consider that a hint is in the paper by Murty and Wong [13]. They showed that if ABC conjecture is true, then largest prime factors of cyclotomic numbers are large enough in a sense.

^{||}Jenkins' program can be downloaded in his web page. See [10].

**It takes about four months for this computation since we used ten CPU's simultaneously.

A.3 Proof of Lemma 3.3

The method of writing proof is according to Hagis and Cohen [7] or Jenkins [9]. Each line begins by listing of the primes that are assumed to be factors of n , and is continued by listing of acceptable values. If there is no acceptable value, then we say “*inadmissible*”. We write p^* to indicate that p is the special prime. Note that two different primes cannot be special simultaneously.

A. $1093 \nmid n$.

A, 1093 : $\Phi_2(1093) = 2 \cdot 547$; $\Phi_3(1093) = 3 \cdot 398581$.

A, 1093*, 547 : $\Phi_3(547) = 3 \cdot 163 \cdot 613$.

A, 1093*, 547, 613 : $\Phi_3(613) = 3 \cdot 7 \cdot 17923$; $\Phi_5(613) = 131 \cdot 20161 \cdot 53551$.

A, 1093*, 547, 613, 17923 : $\Phi_3(17923) = 3 \cdot 13 \cdot 31 \cdot 265717$.

A, 1093*, 547, 613, 17923, 265717 : 265717 is inadmissible.

A, 1093*, 547, 613, 20161 : 20161 is inadmissible.

A, 1093, 398581 : $\Phi_2(398581) = 2 \cdot 17 \cdot 19 \cdot 617$; $\Phi_3(398581) = 3 \cdot 1621 \cdot 32668561$.

A, 1093, 398581*, 617 : $\Phi_3(617) = 97 \cdot 3931$.

A, 1093, 398581*, 617, 3931 : $\Phi_3(3931) = 3 \cdot 7 \cdot 31 \cdot 23743$.

A, 1093, 398581*, 617, 3931, 23743 : $\Phi_3(23743) = 3 \cdot 37 \cdot 5078863$.

A, 1093, 398581*, 617, 3931, 23743, 5078863 : 5078863 is inadmissible.

A, 1093, 398581, 32668561 : $\Phi_2(32668561) = 2 \cdot 19 \cdot 43 \cdot 19993$.

A, 1093, 398581, 32668561*, 19993 : $\Phi_3(19993) = 3 \cdot 73 \cdot 1825297$.

A, 1093, 398581, 32668561*, 19993, 1825297 : $\Phi_3(1825297) = 3 \cdot 326863 \cdot 3397663$.

A, 1093, 398581, 32668561*, 19993, 1825297, 326863 : $\Phi_3(326863) = 3 \cdot 67 \cdot 3313 \cdot 160441$.

A, 1093, 398581, 32668561*, 19993, 1825297, 326863, 3313 : $\Phi_3(3313) = 3 \cdot 7 \cdot 7 \cdot 19 \cdot 3931$.

A, 1093, 398581, 32668561*, 19993, 1825297, 326863, 3313, 3931 : $\Phi_3(3931) = 3 \cdot 7 \cdot 31 \cdot 23743$.

A, 1093, 398581, 32668561*, 19993, 1825297, 326863, 3313, 3931, 23743 : $\Phi_3(23743) = 3 \cdot 37 \cdot 5078863$.

A, 1093, 398581, 32668561*, 19993, 1825297, 326863, 3313, 3931, 23743, 5078863 : 5078863 is inadmissible.

B. $151 \nmid n$.

B, 151 : $\Phi_3(151) = 3 \cdot 7 \cdot 1093$.

B, 151, 1093 : contradiction to A.

C. $31 \nmid n$.

C, 31 : $\Phi_3(31) = 3 \cdot 331$; $\Phi_5(31) = 5 \cdot 11 \cdot 17351$; $\Phi_{13}(31) = 42407 \cdot 2426789 \cdot 7908811$.

C, 31, 331 : $\Phi_3(331) = 3 \cdot 7 \cdot 5233$; $\Phi_5(331) = 5 \cdot 37861 \cdot 63601$.

C, 31, 331, 5233 : $\Phi_2(5233) = 2 \cdot 2617$; $\Phi_3(5233) = 3 \cdot 7 \cdot 31 \cdot 42073$, $\Phi_5(5233) = 2351 \cdot 7741 \cdot 41213191$.

C, 31, 331, 5233*, 2617 : $\Phi_3(2617) = 3 \cdot 193 \cdot 11833$.

C, 31, 331, 5233*, 2617, 11833 : $\Phi_3(11833) = 3 \cdot 13 \cdot 199 \cdot 18043$.

C, 31, 331, 5233*, 2617, 11833, 18043 : $\Phi_3(18043) = 3 \cdot 7 \cdot 15503233$.

C, 31, 331, 5233*, 2617, 11833, 18043, 15503233 : 15503233 is inadmissible.

C, 31, 331, 5233, 42073 : $\Phi_2(42073) = 2 \cdot 109 \cdot 193$; $\Phi_3(42073) = 3 \cdot 19 \cdot 409 \cdot 75931$.

C, 31, 331, 5233, 42073*, 193 : $\Phi_3(193) = 3 \cdot 7 \cdot 1783$.

C, 31, 331, 5233, 42073*, 193, 1783 : $\Phi_3(1783) = 3 \cdot 829 \cdot 1279$; $\Phi_5(1783) = 31 \cdot 67271 \cdot 4849081$.

C, 31, 331, 5233, 42073*, 193, 1783, 1279 : $\Phi_3(1279) = 3 \cdot 229 \cdot 2383$; $\Phi_7(1279) = 56701 \cdot 3745631 \cdot 20627531$.

C, 31, 331, 5233, 42073*, 193, 1783, 1279, 2383 : $\Phi_3(2383) = 3 \cdot 151 \cdot 12541$;

$\Phi_7(2383) = 475637 \cdot 6770429 \cdot 56889841$.

C, 31, 331, 5233, 42073*, 193, 1783, 1279, 2383, 151 : contradiction to B.

C, 31, 331, 5233, 42073*, 193, 1783, 1279, 2383, 475637 : 475637 is inadmissible.

C, 31, 331, 5233, 42073*, 193, 1783, 1279, 20627531 : 20627531 is inadmissible.

C, 31, 331, 5233, 42073*, 193, 1783, 4849081 : 4849081 is inadmissible.

C, 31, 331, 5233, 42073, 75931 : $\Phi_3(75931) = 3 \cdot 7^2 \cdot 19 \cdot 43 \cdot 61 \cdot 787$.

C, 31, 331, 5233, 42073, 75931, 787 : $\Phi_3(787) = 3 \cdot 37^2 \cdot 151$; $\Phi_5(787) = 570821 \cdot 672901$.

C, 31, 331, 5233, 42073, 75931, 787, 151 : contradiction to B.

C, 31, 331, 5233, 42073, 75931, 787, 672901 : $\Phi_2(672901) = 2 \cdot 157 \cdot 2143$.

C, 31, 331, 5233, 42073, 75931, 787, 672901*, 2143 : $\Phi_3(2143) = 3 \cdot 43 \cdot 35617$.

C, 31, 331, 5233, 42073, 75931, 787, 672901*, 2143, 35617 : $\Phi_3(35617) = 3 \cdot 19 \cdot 22256251$.

C, 31, 331, 5233, 42073, 75931, 787, 672901*, 2143, 35617, 22256251 : $\Phi_3(22256251) = 3 \cdot 79 \cdot 397 \cdot 15667 \cdot 336031$.

C, 31, 331, 5233, 42073, 75931, 787, 672901*, 2143, 35617, 22256251, 336031 : 336031 is inadmissible.

C, 31, 331, 5233, 41213191 : $\Phi_3(41213191) = 3 \cdot 199 \cdot 3217 \cdot 4831 \cdot 183067$.

C, 31, 331, 5233, 41213191, 183067 : $\Phi_3(183067) = 3 \cdot 1117 \cdot 10001107$.

C, 31, 331, 5233, 41213191, 183067, 10001107 : $\Phi_3(10001107) = 3 \cdot 7 \cdot 79 \cdot 42409 \cdot 1421647$.

C, 31, 331, 5233, 41213191, 183067, 10001107, 42409 : $\Phi_2(42409) = 2 \cdot 5 \cdot 4241$;
 $\Phi_3(42409) = 3 \cdot 13 \cdot 223 \cdot 206803$; $\Phi_5(42409) = 11 \cdot 491 \cdot 3571 \cdot 12611 \cdot 13299301$.

C, 31, 331, 5233, 41213191, 183067, 10001107, 42409*, 4241 : $\Phi_3(4241) = 13 \cdot 31 \cdot 44641$;
 $\Phi_7(4241) = 29 \cdot 197 \cdot 137957 \cdot 463303 \cdot 15938189$.

C, 31, 331, 5233, 41213191, 183067, 10001107, 42409*, 4241, 44641 : $\Phi_3(44641) = 3 \cdot 7 \cdot 94898263$.

C, 31, 331, 5233, 41213191, 183067, 10001107, 42409*, 4241, 44641, 94898263 : 94898263 is inadmissible.

C, 31, 331, 5233, 41213191, 183067, 10001107, 42409*, 4241, 463303 : $\Phi_3(463303) = 3 \cdot 13 \cdot 19 \cdot 37 \cdot 37 \cdot 211597$.

C, 31, 331, 5233, 41213191, 183067, 10001107, 42409*, 4241, 463303, 211597 : 211597 is inadmissible.

C, 31, 331, 5233, 41213191, 183067, 10001107, 42409, 206803 : $\Phi_3(206803) = 3 \cdot 7 \cdot 19 \cdot 37 \cdot 61 \cdot 47491$.

C, 31, 331, 5233, 41213191, 183067, 10001107, 42409, 206803, 47491 : $\Phi_3(47491) = 3 \cdot 163 \cdot 1063 \cdot 4339$.

C, 31, 331, 5233, 41213191, 183067, 10001107, 42409, 206803, 47491, 1063 : $\Phi_3(1063) = 3 \cdot 377011$;
 $\Phi_7(1063) = 337 \cdot 2423 \cdot 1289513 \cdot 1371511$.

C, 31, 331, 5233, 41213191, 183067, 10001107, 42409, 206803, 47491, 1063, 377011 : 377011 is inadmissible.

C, 31, 331, 5233, 41213191, 183067, 10001107, 42409, 206803, 47491, 1063, 1371511 :
 $\Phi_3(1371511) = 3 \cdot 1579 \cdot 1759 \cdot 225751$.

C, 31, 331, 5233, 41213191, 183067, 10001107, 42409, 206803, 47491, 1063, 1371511, 225751 :
 $\Phi_3(225751) = 3 \cdot 5443 \cdot 3121057$.

C, 31, 331, 5233, 41213191, 183067, 10001107, 42409, 206803, 47491, 1063, 1371511, 225751, 5443 :
 $\Phi_3(5443) = 3 \cdot 7 \cdot 13 \cdot 108541$.

C, 31, 331, 5233, 41213191, 183067, 10001107, 42409, 206803, 47491, 1063, 1371511, 225751, 5443, 108541 :
 $\Phi_2(108541) = 2 \cdot 7 \cdot 7753$.

C, 31, 331, 5233, 41213191, 183067, 10001107, 42409, 206803, 47491, 1063, 1371511, 225751, 5443, 108541*,
7753 : $\Phi_3(7753) = 3 \cdot 7 \cdot 2862703$.

C, 31, 331, 5233, 41213191, 183067, 10001107, 42409, 206803, 47491, 1063, 1371511, 225751, 5443, 108541*,
7753, 2862703 : 2862703 is inadmissible.

C, 31, 331, 5233, 41213191, 183067, 10001107, 42409, 13299301 : $\Phi_2(13299301) = 2 \cdot 283 \cdot 23497$.

C, 31, 331, 5233, 41213191, 183067, 10001107, 42409, 13299301*, 23497 : $\Phi_3(23497) = 3 \cdot 73 \cdot 2521153$.

C, 31, 331, 5233, 41213191, 183067, 10001107, 42409, 13299301*, 23497, 2521153 : 2521153 is inadmissible.

C, 31, 331, 63601 : $\Phi_2(63601) = 2 \cdot 7^2 \cdot 11 \cdot 59$; $\Phi_3(63601) = 3 \cdot 2203 \cdot 612067$;
 $\Phi_5(63601) = 5 \cdot 41 \cdot 271 \cdot 1381 \cdot 4231 \cdot 50408381$.

C, 31, 331, 63601*, 37861 : $\Phi_3(37861) = 3 \cdot 37 \cdot 1201 \cdot 10753$.

(Remark: 37861 is the divisor of $\Phi_5(331)$ which appeared in the third line of C. We often use this method.)

C, 31, 331, 63601*, 37861, 10753 : $\Phi_3(10753) = 3 \cdot 151 \cdot 397 \cdot 643$.

C, 31, 331, 63601*, 37861, 10753, 151 : contradiction to B.

C, 31, 331, 63601, 612067 : $\Phi_3(612067) = 3 \cdot 4801 \cdot 26010319$.
 C, 31, 331, 63601, 612067, 26010319 : 26010319 is inadmissible.
 C, 31, 331, 63601, 50408381 : $\Phi_2(50408381) = 2 \cdot 3 \cdot 8401397$.
 C, 31, 331, 63601, 50408381*, 8401397 : $\Phi_3(8401397) = 7 \cdot 31 \cdot 3943 \cdot 82492897$.
 C, 31, 331, 63601, 50408381*, 8401397, 82492897 : 82492897 is inadmissible.
 C, 31, 17351 : $\Phi_3(17351) = 13 \cdot 1063 \cdot 21787$.
 C, 31, 17351, 21787 : $\Phi_3(21787) = 3 \cdot 31 \cdot 5104249$.
 C, 31, 17351, 21787, 5104249 : $\Phi_2(5104249) = 2 \cdot 5^3 \cdot 17 \cdot 1201$; $\Phi_3(5104249) = 3 \cdot 61 \cdot 216781 \cdot 656737$.
 C, 31, 17351, 21787, 5104249*, 1201 : $\Phi_3(1201) = 3 \cdot 7 \cdot 68743$.
 C, 31, 17351, 21787, 5104249*, 1201, 68743 : 68743 is inadmissible.
 C, 31, 17351, 21787, 5104249, 656737 : $\Phi_2(656737) = 2 \cdot 41 \cdot 8009$; $\Phi_3(656737) = 3 \cdot 7 \cdot 13 \cdot 661 \cdot 787 \cdot 3037$.
 C, 31, 17351, 21787, 5104249, 656737*, 8009 : $\Phi_3(8009) = 6661 \cdot 9631$;
 $\Phi_7(8009) = 7 \cdot 43 \cdot 127 \cdot 491 \cdot 127247 \cdot 305873 \cdot 361313$.
 C, 31, 17351, 21787, 5104249, 656737*, 8009, 9631 : $\Phi_3(9631) = 3 \cdot 151 \cdot 204781$.
 C, 31, 17351, 21787, 5104249, 656737*, 8009, 9631, 151 : contradiction to B.
 C, 31, 17351, 21787, 5104249, 656737*, 8009, 305873 : 305873 is inadmissible.
 C, 31, 17351, 21787, 5104249, 656737, 787 : $\Phi_3(787) = 3 \cdot 37 \cdot 37 \cdot 151$; $\Phi_5(787) = 570821 \cdot 672901$.
 C, 31, 17351, 21787, 5104249, 656737, 787, 151 : contradiction to B.
 C, 31, 17351, 21787, 5104249, 656737, 787, 570821 : $\Phi_2(570821) = 2 \cdot 3 \cdot 7 \cdot 13591$.
 C, 31, 17351, 21787, 5104249, 656737, 787, 570821*, 13591 : $\Phi_3(13591) = 3 \cdot 7 \cdot 7 \cdot 1256659$.
 C, 31, 17351, 21787, 5104249, 656737, 787, 570821*, 13591, 1256659 : $\Phi_3(1256659) = 3 \cdot 8599 \cdot 61216153$.
 C, 31, 17351, 21787, 5104249, 656737, 787, 570821*, 13591, 1256659, 61216153 : 61216153 is inadmissible.
 C, 31, 42407 : $\Phi_3(42407) = 1471 \cdot 1222567$.
 C, 31, 42407, 1222567 : 1222567 is inadmissible.

D. $127 \nmid n$.
 D, 127 : $\Phi_3(127) = 3 \cdot 5419$; $\Phi_7(127) = 7 \cdot 43 \cdot 86353 \cdot 162709$.
 D, 127, 5419 : $\Phi_3(5419) = 3 \cdot 31 \cdot 313 \cdot 1009$.
 D, 127, 5419, 31 : contradiction to C.
 D, 127, 162709 : $\Phi_2(162709) = 2 \cdot 5 \cdot 53 \cdot 307$.
 D, 127, 162709*, 86353 : $\Phi_5(86353) = 11 \cdot 281 \cdot 1021 \cdot 1964041 \cdot 8970971$.
 D, 127, 162709*, 86353, 8970971 : 8970971 is inadmissible.

E. $19 \nmid n$.
 E, 19 : $\Phi_3(19) = 3 \cdot 127$; $\Phi_5(19) = 151 \cdot 911$; $\Phi_7(19) = 701 \cdot 70841$; $\Phi_{11}(19) = 104281 \cdot 62060021$.
 E, 19, 127 : contradiction to D.
 E, 19, 151 : contradiction to B.
 E, 19, 70841 : $\Phi_2(70841) = 2 \cdot 3 \cdot 11807$; $\Phi_3(70841) = 39103 \cdot 128341$.
 E, 19, 70841*, 701 : $\Phi_3(701) = 492103$.
 E, 19, 70841*, 701, 492103 : $\Phi_3(492103) = 3 \cdot 307 \cdot 1609 \cdot 163417$.
 E, 19, 70841*, 701, 492103, 163417 : $\Phi_3(163417) = 3 \cdot 7 \cdot 463 \cdot 2746609$.
 E, 19, 70841*, 701, 492103, 163417, 2746609 : $\Phi_3(2746609) = 3 \cdot 19 \cdot 127 \cdot 1879 \cdot 554611$.
 E, 19, 70841*, 701, 492103, 163417, 2746609, 127 : contradiction to D.
 E, 19, 70841, 39103 : $\Phi_3(39103) = 3 \cdot 7561 \cdot 67411$.
 E, 19, 70841, 39103, 67411 : 67411 is inadmissible.
 E, 19, 62060021 : $\Phi_2(62060021) = 2 \cdot 3^{247} \cdot 109 \cdot 673$.
 E, 19, 62060021*, 104281 : $\Phi_3(104281) = 3 \cdot 7 \cdot 43 \cdot 67 \cdot 179743$.
 E, 19, 62060021*, 104281, 179743 : $\Phi_3(179743) = 3 \cdot 7 \cdot 31 \cdot 49627843$.

E, 19, 62060021*, 104281, 179743, 31 : contradiction to C.

F, 11 $\nmid n$.

F, 11 : $\Phi_3(11) = 7 \cdot 19$; $\Phi_5(11) = 5 \cdot 3221$; $\Phi_7(11) = 43 \cdot 45319$; $\Phi_{11}(11) = 15797 \cdot 1806113$.

F, 11, 19 : contradiction to E.

F, 11, 3221 : $\Phi_2(3221) = 2 \cdot 3^2 \cdot 179$; $\Phi_3(3221) = 10378063$; $\Phi_7(3221) = 7 \cdot 673 \cdot 10333 \cdot 248879 \cdot 92204351$.

F, 11, 3221*, 179 : $\Phi_3(179) = 7 \cdot 4603$; $\Phi_5(179) = 11 \cdot 93853931$.

F, 11, 3221*, 179, 4603 : $\Phi_3(4603) = 3 \cdot 7 \cdot 1009153$; $\Phi_3(4603) = 11 \cdot 911 \cdot 208511 \cdot 214891$.

F, 11, 3221*, 179, 4603, 1009153 : 1009153 is inadmissible.

F, 11, 3221*, 179, 4603, 208511 : $\Phi_3(208511) = 7 \cdot 31 \cdot 61 \cdot 661 \cdot 4969$.

F, 11, 3221*, 179, 4603, 208511, 31 : contradiction to C.

F, 11, 3221*, 179, 93853931 : $\Phi_3(93853931) = 19^2 \cdot 31 \cdot 151 \cdot 8293 \cdot 628561$.

F, 11, 3221*, 179, 93853931, 151 : contradiction to B.

F, 11, 3221, 10378063 : 10378063 is inadmissible.

F, 11, 3221, 248879 : 248879 is inadmissible.

F, 11, 45319 : $\Phi_3(45319) = 3 \cdot 127 \cdot 5390701$.

F, 11, 45319, 127 : contradiction to D.

F, 11, 1806113 : $\Phi_2(1806113) = 2 \cdot 3 \cdot 17 \cdot 17707$.

F, 11, 1806113*, 17707 : $\Phi_3(17707) = 3 \cdot 7^2 \cdot 2133031$.

F, 11, 1806113*, 17707, 2133031 : 2133031 is inadmissible.

G, 7 $\nmid n$.

G, 7 : $\Phi_3(7) = 3 \cdot 19$; $\Phi_5(7) = 2801$; $\Phi_7(7) = 29 \cdot 4733$; $\Phi_{11}(7) = 1123 \cdot 293459$.

G, 7, 19 : contradiction to E.

G, 7, 2801 : $\Phi_2(2801) = 2 \cdot 3 \cdot 467$; $\Phi_3(2801) = 37 \cdot 43 \cdot 4933$; $\Phi_5(2801) = 5 \cdot 1956611 \cdot 6294091$.

G, 7, 2801*, 467 : $\Phi_3(467) = 19 \cdot 11503$; $\Phi_5(467) = 11 \cdot 31 \cdot 41 \cdot 3409261$.

G, 7, 2801*, 467, 19 : contradiction to E.

G, 7, 2801*, 467, 31 : contradiction to C.

G, 7, 2801, 4933 : $\Phi_2(4933) = 2 \cdot 2467$; $\Phi_3(4933) = 3 \cdot 127 \cdot 193 \cdot 331$.

G, 7, 2801, 4933*, 2467 : $\Phi_3(2467) = 3 \cdot 271 \cdot 7489$.

G, 7, 2801, 4933*, 2467, 271 : $\Phi_3(271) = 3 \cdot 24571$; $\Phi_5(271) = 5 \cdot 251 \cdot 4313591$; $\Phi_7(271) = 9170197 \cdot 43355341$.

G, 7, 2801, 4933*, 2467, 271, 24571 : 24571 is inadmissible.

G, 7, 2801, 4933*, 2467, 271, 4313591 : 4313591 is inadmissible.

G, 7, 2801, 4933*, 2467, 271, 9170197 : 9170197 is inadmissible.

G, 7, 2801, 4933, 127 : contradiction to D.

G, 7, 2801, 6294091 : 6294091 is inadmissible.

G, 7, 4733 : $\Phi_2(4733) = 2 \cdot 3^2 \cdot 263$; $\Phi_3(4733) = 22406023$.

G, 7, 4733*, 263 : $\Phi_3(263) = 7^2 \cdot 13 \cdot 109$.

G, 7, 4733*, 263, 109 : $\Phi_3(109) = 3 \cdot 7 \cdot 571$; $\Phi_5(109) = 31 \cdot 191 \cdot 24061$; $\Phi_7(109) = 113 \cdot 281 \cdot 53306107$.

G, 7, 4733*, 263, 109, 571 : $\Phi_3(571) = 3 \cdot 7 \cdot 103 \cdot 151$; $\Phi_5(571) = 5 \cdot 1831 \cdot 11631811$.

G, 7, 4733*, 263, 109, 571, 151 : contradiction to B.

G, 7, 4733*, 263, 109, 571, 11631811 : $\Phi_3(11631811) = 3 \cdot 7 \cdot 13 \cdot 19 \cdot 37^2 \cdot 73 \cdot 211 \cdot 1237$.

G, 7, 4733*, 263, 109, 571, 11631811, 19 : contradiction to E.

G, 7, 4733*, 263, 109, 31 : contradiction to C.

G, 7, 4733*, 263, 109, 53306107 : 53306107 is inadmissible.

G, 7, 4733, 22406023 : 22406023 is inadmissible.

G, 7, 293459 : 293459 is inadmissible.

H. 23 $\nmid n$.

H, 23 : $\Phi_3(23) = 7 \cdot 79$; $\Phi_5(23) = 292561$; $\Phi_7(23) = 29 \cdot 5336717$.

H, 23, 7 : contradiction to G.

H, 23, 292561 : $\Phi_2(292561) = 2 \cdot 19 \cdot 7699$.

H, 23, 292561*, 19 : contradiction to E.

H, 23, 5336717 : $\Phi_2(5336717) = 2 \cdot 3 \cdot 889453$.

H, 23, 5336717*, 889453 : 889453 is inadmissible.

I. 131 $\nmid n$.

I, 131 : $\Phi_3(131) = 17293$; $\Phi_5(131) = 5 \cdot 61 \cdot 973001$; $\Phi_7(131) = 127 \cdot 189967 \cdot 211093$.

I, 131, 17293 : $\Phi_2(17293) = 2 \cdot 8647$; $\Phi_3(17293) = 3 \cdot 13 \cdot 7668337$.

I, 131, 17293*, 8647 : $\Phi_3(8647) = 3 \cdot 7 \cdot 37 \cdot 157 \cdot 613$.

I, 131, 17293*, 8647, 7 : contradiction to G.

I, 131, 17293, 7668337 : $\Phi_2(7668337) = 2 \cdot 23 \cdot 166703$; $\Phi_3(7668337) = 3 \cdot 801337 \cdot 24460537$.

I, 131, 17293, 7668337*, 23 : contradiction to H.

I, 131, 17293, 7668337, 801337 : $\Phi_2(801337) = 2 \cdot 59 \cdot 6791$.

I, 131, 17293, 7668337, 801337*, 24460537 : 24460537 is inadmissible.

I, 131, 973001 : $\Phi_2(973001) = 2 \cdot 3 \cdot 257 \cdot 631$.

I, 131, 973001*, 257 : $\Phi_3(257) = 61 \cdot 1087$.

I, 131, 973001*, 257, 1087 : $\Phi_3(1087) = 3 \cdot 7 \cdot 199 \cdot 283$.

I, 131, 973001*, 257, 1087, 7 : contradiction to G.

I, 131, 127 : contradiction to D.

J. 37 $\nmid n$.

J, 37 : $\Phi_2(37) = 2 \cdot 19$; $\Phi_3(37) = 3 \cdot 7 \cdot 67$; $\Phi_5(37) = 11 \cdot 41 \cdot 4271$; $\Phi_7(37) = 71 \cdot 37140797$.

J, 37*, 19 : contradiction to E

J, 37, 7 : contradiction to G.

J, 37, 11 : contradiction to F.

J, 37, 37140797 : $\Phi_2(37140797) = 2 \cdot 3 \cdot 347 \cdot 17839$; $\Phi_3(37140797) = 1609 \cdot 3469 \cdot 4831 \cdot 51157$.

J, 37, 37140797*, 17839 : $\Phi_3(17839) = 3 \cdot 13 \cdot 8160199$.

J, 37, 37140797*, 17839, 8160199 : 8160199 is inadmissible.

J, 37, 37140797, 3469 : $\Phi_2(3469) = 2 \cdot 5 \cdot 347$; $\Phi_3(3469) = 3 \cdot 7 \cdot 19 \cdot 30169$.

J, 37, 37140797, 3469*, 1609 : $\Phi_3(1609) = 3 \cdot 863497$.

J, 37, 37140797, 3469*, 1609, 863497 : 863497 is inadmissible.

J, 37, 37140797, 3469, 19 : contradiction to E.

K. 61 $\nmid n$.

K, 61 : $\Phi_2(61) = 2 \cdot 31$; $\Phi_3(61) = 3 \cdot 13 \cdot 97$; $\Phi_5(61) = 5 \cdot 131 \cdot 21491$.

K, 61*, 31 : contradiction to C.

K, 61, 97 : $\Phi_2(97) = 2 \cdot 7^2$; $\Phi_3(97) = 3 \cdot 3169$; $\Phi_5(97) = 11 \cdot 31 \cdot 262321$; $\Phi_7(97) = 43 \cdot 967 \cdot 20241187$.

K, 61, 97*, 7 : contradiction to G.

K, 61, 97, 3169 : $\Phi_2(3169) = 2 \cdot 5 \cdot 317$; $\Phi_3(3169) = 3 \cdot 3348577$.

K, 61, 97, 3169*, 317 : $\Phi_3(317) = 7 \cdot 14401$; $\Phi_5(317) = 11 \cdot 311 \cdot 2961121$.

K, 61, 97, 3169*, 317, 7 : contradiction to G.

K, 61, 97, 3169*, 317, 11 : contradiction to F.

K, 61, 97, 3169, 3348577 : $\Phi_2(3348577) = 2 \cdot 1674289$.

K, 61, 97, 3169, 3348577*, 1674289 : 1674289 is inadmissible.

K, 61, 97, 11 : contradiction to F.

K, 61, 97, 20241187 : 20241187 is inadmissible.

K, 61, 131 : contradiction to I.

L, 13 $\nmid n$.

L, 13 : $\Phi_2(13) = 2 \cdot 7$; $\Phi_3(13) = 3 \cdot 61$; $\Phi_5(13) = 30941$; $\Phi_7(13) = 5229043$;

$\Phi_{11}(13) = 23 \cdot 419 \cdot 859 \cdot 18041$; $\Phi_{13}(13) = 53 \cdot 264031 \cdot 1803647$.

L, 13*, 7 : contradiction to G.

L, 13, 61 : contradiction to K.

L, 13, 30941 : $\Phi_2(30941) = 2 \cdot 3^4 191$; $\Phi_3(30941) = 157 \cdot 433 \cdot 14083$.

L, 13, 30941*, 191 : $\Phi_3(191) = 7 \cdot 13^2 31$; $\Phi_5(191) = 5 \cdot 11 \cdot 1871 \cdot 13001$;

$\Phi_7(191) = 127 \cdot 197 \cdot 10627 \cdot 183569$; $\Phi_{13}(191) = 131 \cdot 1483 \cdot 9049 \cdot 92041 \cdot 301627 \cdot 48552947$.

L, 13, 30941*, 191, 7 : contradiction to G.

L, 13, 30941*, 191, 11 : contradiction to F.

L, 13, 30941*, 191, 127 : contradiction to D.

L, 13, 30941*, 191, 131 : contradiction to I.

L, 13, 30941, 433 : $\Phi_2(433) = 2 \cdot 7 \cdot 31$; $\Phi_3(433) = 3 \cdot 37 \cdot 1693$; $\Phi_5(433) = 11 \cdot 1811 \cdot 1768661$.

L, 13, 30941, 433*, 31 : contradiction to C.

L, 13, 30941, 433, 37 : contradiction to J.

L, 13, 30941, 433, 11 : contradiction to F.

L, 13, 5229043 : $\Phi_3(5229043) = 3 \cdot 31 \cdot 4051 \cdot 72577051$.

L, 13, 5229043, 31 : contradiction to C.

L, 13, 23 : contradiction to H.

L, 13, 1803647 : 1803647 is inadmissible.

M, 3 $\nmid n$.

M, 3 : $\Phi_3(3) = 13$; $\Phi_5(3) = 11^2$; $\Phi_7(3) = 1093$; $\Phi_{11}(3) = 23 \cdot 3851$; $\Phi_{13}(11) = 797161$;

$\Phi_{17}(3) = 1871 \cdot 34511$; $\Phi_{19}(3) = 1597 \cdot 363889$; $\Phi_{29}(3) = 59 \cdot 28537 \cdot 20381027$;

$\Phi_{31}(3) = 683 \cdot 102673 \cdot 4404047$; $\Phi_{47}(3) = 1223 \cdot 21997 \cdot 5112661 \cdot 96656723$.

M, 3, 13 : contradiction to L.

M, 3, 11 : contradiction to F.

M, 3, 1093 : contradiction to A.

M, 3, 23 : contradiction to H.

M, 3, 797161 : $\Phi_2(797161) = 2 \cdot 398581$; $\Phi_3(797161) = 3 \cdot 61 \cdot 151 \cdot 22996651$.

M, 3, 797161*, 398581 : $\Phi_3(398581) = 3 \cdot 1621 \cdot 32668561$.

M, 3, 797161*, 398581, 32668561 : 32668561 is inadmissible.

M, 3, 797161, 151 : contradiction to B.

M, 3, 1871 : $\Phi_3(1871) = 7 \cdot 157 \cdot 3187$.

M, 3, 1871, 7 : contradiction to G.

M, 3, 1597 : $\Phi_2(1597) = 2 \cdot 17 \cdot 47$; $\Phi_3(1597) = 3 \cdot 43 \cdot 73 \cdot 271$.

M, 3, 1597*, 47 : $\Phi_3(47) = 37 \cdot 61$; $\Phi_5(47) = 11 \cdot 31 \cdot 14621$; $\Phi_{13}(47) = 53 \cdot 2237 \cdot 14050609 \cdot 71265169$.

M, 3, 1597*, 47, 37 : contradiction to J.

M, 3, 1597*, 47, 31 : contradiction to C.

M, 3, 1597*, 47, 14050609 : 14050609 is inadmissible.

M, 3, 1597, 271 : $\Phi_3(271) = 3 \cdot 24571$; $\Phi_5(271) = 5 \cdot 251 \cdot 4313591$; $\Phi_7(271) = 9170197 \cdot 43355341$.

M, 3, 1597, 271, 24571 : 24571 is inadmissible.

M, 3, 1597, 271, 4313591 : 4313591 is inadmissible.

M, 3, 1597, 271, 9170197 : $\Phi_2(9170197) = 2 \cdot 19 \cdot 241321$.

M, 3, 1597, 271, 9170197, 19 : contradiction to E.

M, 3, 20381027 : $\Phi_3(20381027) = 7 \cdot 67 \cdot 19687 \cdot 44988319$.
M, 3, 20381027, 7 : contradiction to G.
M, 3, 683 : $\Phi_3(683) = 7 \cdot 66739$.
M, 3, 683, 7 : contradiction to G.
M, 3, 96656723 : 96656723 is inadmissible.

N. 5 $\nmid n$.
N, 5 : $\Phi_2(5) = 2 \cdot 3$; $\Phi_3(5) = 31$; $\Phi_5(5) = 11 \cdot 71$; $\Phi_7(5) = 19531$; $\Phi_{11}(5) = 12207031$;
 $\Phi_{11}(5) = 191 \cdot 6271 \cdot 3981071$.
N, 5*, 3 : contradiction to M.
N, 5, 31 : contradiction to C.
N, 5, 11 : contradiction to F.
N, 5, 19531 : $\Phi_5(19531) = 5 \cdot 191 \cdot 4760281 \cdot 32009891$.
N, 5, 19531, 32009891 : $\Phi_3(32009891) = 7 \cdot 283 \cdot 468913 \cdot 1103041$.
N, 5, 19531, 32009891, 7 : contradiction to G.
N, 5, 12207031 : $\Phi_3(12207031) = 3 \cdot 7 \cdot 1041757 \cdot 6811369$.
N, 5, 12207031, 7 : contradiction to G.
N, 5, 3981071 : 3981071 is inadmissible.

O. 29 $\nmid n$.
O, 29 : $\Phi_2(29) = 2 \cdot 3 \cdot 5$; $\Phi_3(29) = 13 \cdot 67$; $\Phi_5(29) = 732541$; $\Phi_7(29) = 7 \cdot 88009573$.
O, 29*, 3 : contradiction to M.
O, 29, 13 : contradiction to L.
O, 29, 732541 : $\Phi_2(732541) = 2 \cdot 47 \cdot 7793$; $\Phi_3(732541) = 3 \cdot 43 \cdot 271 \cdot 15349897$.
O, 29, 732541*, 7793 : $\Phi_3(7793) = 7 \cdot 8676949$; $\Phi_3(7793) = 11 \cdot 71^2 \cdot 3701 \cdot 17974051$.
O, 29, 732541*, 7793, 7 : contradiction to G.
O, 29, 732541*, 7793, 11 : contradiction to F.
O, 29, 732541, 3 : contradiction to M.
O, 29, 7 : contradiction to G.

P. 43 $\nmid n$.
P, 43 : $\Phi_3(43) = 3 \cdot 631$; $\Phi_5(43) = 3500201$; $\Phi_7(43) = 7 \cdot 5839 \cdot 158341$.
P, 43, 3 : contradiction to M.
P, 43, 3500201 : $\Phi_2(3500201) = 2 \cdot 3 \cdot 583367$; $\Phi_3(3500201) = 13 \cdot 139 \cdot 28411 \cdot 238639$.
P, 43, 3500201*, 3 : contradiction to M.
P, 43, 3500201, 13 : contradiction to L.
P, 43, 7 : contradiction to G.

Q. 1051 $\nmid n$.
Q, 1051 : $\Phi_3(1051) = 3 \cdot 368551$; $\Phi_5(1051) = 5 \cdot 71 \cdot 241 \cdot 14275091$.
Q, 1051, 3 : contradiction to M.
Q, 1051, 5 : contradiction to N.

R. 17 $\nmid n$.
R, 17 : $\Phi_2(17) = 2 \cdot 3^2$; $\Phi_3(17) = 307$; $\Phi_5(17) = 88741$; $\Phi_7(17) = 25646167$.
R, 17*, 3 : contradiction to M.
R, 17, 307; $\Phi_3(307) = 3 \cdot 43 \cdot 733$; $\Phi_5(307) = 1051 \cdot 1621 \cdot 5231$.
R, 17, 307, 3 : contradiction to M.
R, 17, 307, 1051 : contradiction to Q.
R, 17, 88741 : $\Phi_2(88741) = 2 \cdot 44371$; $\Phi_3(88741) = 5 \cdot 71 \cdot 241 \cdot 14275091$.

R, 17, 88741*, 44371 : 44371 is inadmissible.

R, 17, 88741, 5 : contradiction to N.

R, 17, 25646167 : 25646167 is inadmissible.

S, 71 $\nmid n$.

S, 71 : $\Phi_3(71) = 5113$; $\Phi_5(71) = 5 \cdot 11 \cdot 211 \cdot 2221$; $\Phi_7(71) = 7 \cdot 883 \cdot 21020917$.

S, 71, 5113 : $\Phi_2(5113) = 2 \cdot 2557$; $\Phi_3(5113) = 3 \cdot 8715961$.

S, 71, 5113*, 2557 : $\Phi_3(2557) = 3 \cdot 7 \cdot 13^2 \cdot 19 \cdot 97$.

S, 71, 5113*, 2557, 19 : contradiction to E.

S, 71, 5113, 3 : contradiction to M.

S, 71, 11 : contradiction to F.

S, 71, 7 : contradiction to G.

T, 113 $\nmid n$.

T, 113 : $\Phi_2(113) = 2 \cdot 3 \cdot 19$; $\Phi_3(113) = 13 \cdot 991$; $\Phi_5(113) = 11 \cdot 251 \cdot 59581$; $\Phi_7(113) = 7 \cdot 44983 \cdot 6670903$.

T, 113*, 19 : contradiction to E.

T, 113, 13 : contradiction to L.

T, 113, 11 : contradiction to F.

T, 113, 7 : contradiction to G.

U, 197 $\nmid n$.

U, 197 : $\Phi_2(197) = 2 \cdot 3^2 \cdot 11$; $\Phi_3(197) = 19 \cdot 2053$; $\Phi_5(197) = 661 \cdot 991 \cdot 2311$; $\Phi_7(197) = 7 \cdot 29 \cdot 97847 \cdot 2957767$.

U, 197*, 11 : contradiction to F.

U, 197, 19 : contradiction to E.

U, 197, 991 : $\Phi_3(991) = 3 \cdot 7 \cdot 13^2 \cdot 277$.

U, 197, 991, 7 : contradiction to G.

U, 197, 7 : contradiction to G.

V, 211 $\nmid n$.

V, 211 : $\Phi_3(211) = 3 \cdot 13 \cdot 31 \cdot 37$; $\Phi_5(211) = 5 \cdot 1361 \cdot 292661$; $\Phi_7(211) = 7 \cdot 307189 \cdot 41233879$.

V, 211, 31 : contradiction to C.

V, 211, 5 : contradiction to N.

V, 211, 7 : contradiction to G.

W, 239 $\nmid n$.

W, 239 : $\Phi_3(239) = 19 \cdot 3019$; $\Phi_7(239) = 7 \cdot 29 \cdot 245561 \cdot 3754507$.

W, 239, 19 : contradiction to E.

W, 239, 7 : contradiction to G.

X, 281 $\nmid n$.

X, 281 : $\Phi_2(281) = 2 \cdot 3 \cdot 47$; $\Phi_3(281) = 109 \cdot 727$; $\Phi_5(281) = 5 \cdot 31 \cdot 271 \cdot 148961$.

X, 281*, 3 : contradiction to M.

X, 281, 727 : $\Phi_3(727) = 3 \cdot 176419$; $\Phi_5(727) = 14281 \cdot 19587401$.

X, 281, 727, 3 : contradiction to M.

X, 281, 727, 19587401 : $\Phi_2(19587401) = 2 \cdot 3^2 \cdot 311 \cdot 3499$.

X, 281, 727, 19587401*, 3 : contradiction to M.

X, 281, 31 : contradiction to C.

Y, 337 $\nmid n$.

Y, 337 : $\Phi_2(337) = 2 \cdot 13^2$; $\Phi_3(337) = 3 \cdot 43 \cdot 883$.

Y, 337*, 13 : contradiction to L.

Y, 337, 3 : contradiction to M.

Z, 379 $\nmid n$.

Z, 379 : $\Phi_3(379) = 3 \cdot 61 \cdot 787$; $\Phi_5(379) = 11 \cdot 41 \cdot 45869891$.

Z, 379, 61 : contradiction to K.

Z, 379, 11 : contradiction to F.

AA, 421 $\nmid n$.

AA, 421 : $\Phi_2(421) = 2 \cdot 211$; $\Phi_3(421) = 3 \cdot 59221$; $\Phi_5(421) = 5 \cdot 11 \cdot 181 \cdot 191 \cdot 16561$.

AA, 421*, 211 : contradiction to V.

AA, 421, 3 : contradiction to M.

AA, 421, 11 : contradiction to F.

AB, 449 $\nmid n$.

AB, 449 : $\Phi_2(449) = 2 \cdot 3^2 5^2$; $\Phi_3(449) = 97 \cdot 2083$.

AB, 449*, 5 : contradiction to N.

AB, 449, 2083 : $\Phi_3(2083) = 3 \cdot 7 \cdot 13 \cdot 15901$.

AB, 449, 2083, 7 : contradiction to G.

AC, 463 $\nmid n$.

AC, 463 : $\Phi_3(463) = 3 \cdot 19 \cdot 3769$; $\Phi_5(463) = 881 \cdot 52274161$.

AC, 463, 19 : contradiction to E.

AC, 463, 881 : $\Phi_2(881) = 2 \cdot 3^2 7^2$; $\Phi_3(881) = 19 \cdot 40897$; $\Phi_5(881) = 5 \cdot 146521 \cdot 823241$.

AC, 463, 881*, 7 : contradiction to G.

AC, 463, 881, 19 : contradiction to E.

AC, 463, 881, 5 : contradiction to N.

AD, 491 $\nmid n$.

AD, 491 : $\Phi_3(491) = 37 \cdot 6529$; $\Phi_5(491) = 5 \cdot 101 \cdot 191 \cdot 603791$.

AD, 491, 37 : contradiction to J.

AD, 491, 5 : contradiction to N.

AE, 547 $\nmid n$.

AE, 547 : $\Phi_3(547) = 3 \cdot 163 \cdot 613$.

AE, 547, 3 : contradiction to M.

AF, 617 $\nmid n$.

AF, 617 : $\Phi_2(617) = 2 \cdot 3 \cdot 103$; $\Phi_3(617) = 97 \cdot 3931$.

AF, 617*, 3 : contradiction to M.

AF, 617, 3931 : $\Phi_3(3931) = 3 \cdot 7 \cdot 31 \cdot 23743$.

AF, 617, 3931, 31 : contradiction to C.

AG, 631 $\nmid n$.

AG, 631 : $\Phi_3(631) = 3 \cdot 307 \cdot 433$; $\Phi_5(631) = 5 \cdot 11 \cdot 41 \cdot 1511 \cdot 46601$.

AG, 631, 3 : contradiction to M.

AG, 631, 11 : contradiction to F.

AH, 659 $\nmid n$.

AH, 659 : $\Phi_3(659) = 13 \cdot 33457$; $\Phi_5(659) = 31 \cdot 6131 \cdot 993821$.

AH, 659, 13 : contradiction to L.

AH, 659, 31 : contradiction to C.

AI, 673 $\nmid n$.

AI, 673 : $\Phi_2(673) = 2 \cdot 337$; $\Phi_3(673) = 3 \cdot 151201$; $\Phi_5(673) = 421 \cdot 2531 \cdot 192811$.

AI, 673*, 337 : contradiction to Y.

AI, 673, 3 : contradiction to M.

AI, 673, 421 : contradiction to AA.

AJ. 743 $\dagger n$.

AJ, 743 : $\Phi_3(743) = 552793$.

AJ, 743, 552793 : $\Phi_2(552793) = 2 \cdot 11 \cdot 25127$; $\Phi_3(552793) = 3 \cdot 19 \cdot 421 \cdot 12734119$;

$\Phi_5(552793) = 191 \cdot 661 \cdot 1531 \cdot 13931 \cdot 65521 \cdot 529271$.

AJ, 743, 552793*, 11 : contradiction to F.

AJ, 743, 552793, 19 : contradiction to E.

AJ, 743, 552793, 13931 : 13931 is inadmissible.

AK. 757 $\dagger n$.

AK, 757 : $\Phi_2(757) = 2 \cdot 379$; $\Phi_3(757) = 3 \cdot 13 \cdot 14713$; $\Phi_5(757) = 11 \cdot 191 \cdot 2521 \cdot 62081$.

AK, 757*, 379 : contradiction to Z.

AK, 757, 13 : contradiction to L.

AK, 757, 11 : contradiction to F.

AL. 827 $\dagger n$.

AL, 827 : $\Phi_3(827) = 684757$.

AL, 827, 684757 : $\Phi_2(684757) = 2 \cdot 342379$.

AL, 827, 684757*, 342379 : $\Phi_3(342379) = 3 \cdot 7 \cdot 61 \cdot 2803 \cdot 32647$.

AL, 827, 684757*, 342379, 7 : contradiction to G.

AM. 911 $\dagger n$.

AM, 911 : $\Phi_3(911) = 830833$; $\Phi_5(911) = 5 \cdot 11 \cdot 701 \cdot 17884211$.

AM, 911, 830833 : $\Phi_2(830833) = 2 \cdot 127 \cdot 3271$; $\Phi_3(830833) = 3 \cdot 13 \cdot 61 \cdot 337 \cdot 861001$.

AM, 911, 830833*, 127 : contradiction to D.

AM, 911, 830833, 61 : contradiction to K.

AM, 911, 11 : contradiction to F.

AN. 953 $\dagger n$.

AN, 953 : $\Phi_2(953) = 2 \cdot 3^2 53$; $\Phi_3(953) = 181 \cdot 5023$; $\Phi_5(953) = 41 \cdot 1601 \cdot 2161 \cdot 5821$;

$\Phi_7(953) = 7 \cdot 29 \cdot 71 \cdot 113 \cdot 127 \cdot 379 \cdot 9566159$.

AN, 953*, 3 : contradiction to M.

AN, 953, 5023 : $\Phi_3(5023) = 3 \cdot 7 \cdot 19 \cdot 63247$.

AN, 953, 5023, 19 : contradiction to E.

AN, 953, 2161 : $\Phi_2(2161) = 2 \cdot 23 \cdot 47$; $\Phi_3(2161) = 3 \cdot 13 \cdot 119797$.

AN, 953, 2161*, 23 : contradiction to H.

AN, 953, 2161, 13 : contradiction to L.

AN, 953, 7 : contradiction to G.

AO. 967 $\dagger n$.

AO, 967 : $\Phi_3(967) = 3 \cdot 67 \cdot 4657$.

AO, 967, 3 : contradiction to M.

A.4 Proof of Lemma 3.4

In order to prove Lemma 3.4, we need to show that (2.1) does not contain cyclotomic numbers described in §3.3. There are 84 such cyclotomic numbers. The writing method is similar to one of §A.3. In the proof, X is the set given in the statement of Lemma 3.3.

AP. 67 : $\Phi_7(67) = 175897 \cdot 522061$.

AP, 67, 522061 : $\Phi_2(522061) = 2 \cdot 261031$.

AP, 67, 522061*, 261031 : 261031 is inadmissible.

AQ. 173 : $\Phi_7(173) = 3144079 \cdot 8576317$.

AQ, 173, 3144079 : $\Phi_3(3144079) = 3 \cdot 13 \cdot 67 \cdot 13267 \cdot 285151$.

AQ, 173, 3144079, 3 : $3 \in X$.

AR. 271 : $\Phi_7(271) = 9170197 \cdot 43355341$.

AR, 271, 9170197 : $\Phi_2(9170197) = 2 \cdot 19 \cdot 241321$.

AR, 271, 9170197*, 19 : $19 \in X$.

AS. 347 : $\Phi_7(347) = 39577763 \cdot 44236319$.

AS, 347, 39577763 : 39577763 is inadmissible.

AT. 409 : $\Phi_7(409) = 6133 \cdot 15919 \cdot 48063373$.

AT, 409, 48063373 : $\Phi_2(48063373) = 2 \cdot 24031687$.

AT, 409, 48063373*, 24031687 : 24031687 is inadmissible.

AU. 439 : $\Phi_7(439) = 7883 \cdot 63841 \cdot 14255627$.

AU, 439, 14255627 : $\Phi_3(14255627) = 13 \cdot 67 \cdot 811 \cdot 10501 \cdot 27397$.

AU, 439, 14255627, 13 : $13 \in X$.

AV. 607 : $\Phi_7(607) = 54517 \cdot 415759 \cdot 2210419$.

AV, 607, 415759 : 415759 is inadmissible.

AW. 619 : $\Phi_7(619) = 3389 \cdot 3732919 \cdot 4453751$.

AW, 619, 3732919 : $\Phi_3(3732919) = 3 \cdot 37 \cdot 73 \cdot 3673 \cdot 468199$.

AW, 619, 3732919, 3 : $3 \in X$.

AX. 653 : $\Phi_7(653) = 21757 \cdot 706763 \cdot 5049773$.

AX, 653, 21757 : $\Phi_2(21757) = 2 \cdot 11 \cdot 23 \cdot 43$.

AX, 653, 21757*, 11 : $11 \in X$.

AY. 853 : $\Phi_7(853) = 2647 \cdot 11824121 \cdot 12321989$.

AY, 853, 11824121 : $\Phi_2(11824121) = 2 \cdot 3 \cdot 1381 \cdot 1427$.

AY, 853, 11824121*, 3 : $3 \in X$.

AZ. 887 : $\Phi_7(887) = 5167 \cdot 6271651 \cdot 15045661$.

AZ, 887, 5167 : $\Phi_3(5167) = 3 \cdot 8901019$.

AZ, 887, 5167, 3 : $3 \in X$.

BA. 1279 : $\Phi_7(1279) = 56701 \cdot 3745631 \cdot 20627531$.

BA, 1279, 3745631 : $\Phi_3(3745631) = 31 \cdot 37 \cdot 73063 \cdot 167413$.

BA, 1279, 3745631, 31 : $31 \in X$.

BB. 1451 : $\Phi_7(1451) = 2381 \cdot 52584967 \cdot 74590391$.

BB, 1451, 52584967 : 52584967 is inadmissible.

BC. 2383 : $\Phi_7(2383) = 475637 \cdot 6770429 \cdot 56889841$.

BC, 2383, 475637 : $\Phi_2(475637) = 2 \cdot 3 \cdot 79273$.
 BC, 2383, 475637*, 3 : $3 \in X$.

 BD, 3089 : $\Phi_7(3089) = 1303 \cdot 89237 \cdot 316793 \cdot 23592997$.
 BD, 3089, 89237 : $\Phi_2(89237) = 2 \cdot 3 \cdot 107 \cdot 139$.
 BD, 3089, 89237*, 3 : $3 \in X$.

 BE, 4129 : $\Phi_7(4129) = 5867 \cdot 17053 \cdot 714463 \cdot 69339047$.
 BE, 4129, 69339047 : 69339047 is inadmissible.

 BF, 4289 : $\Phi_7(4289) = 1471 \cdot 8807 \cdot 9619 \cdot 32467 \cdot 1538951$.
 BF, 4289, 9619 : $\Phi_3(9619) = 3 \cdot 30844927$.
 BF, 4289, 9619, 3 : $3 \in X$.

 BG, 5399 : $\Phi_7(5399) = 2731 \cdot 9941 \cdot 14811889 \cdot 61602479$.
 BG, 5399, 2731 : $\Phi_3(2731) = 3 \cdot 61 \cdot 40771$.
 BG, 5399, 2731, 3 : $3 \in X$.

 BH, 5689 : $\Phi_7(5689) = 9479 \cdot 107941 \cdot 338773 \cdot 97821473$.
 BH, 5689, 107941 : $\Phi_2(107941) = 2 \cdot 31 \cdot 1741$; $\Phi_3(107941) = 3 \cdot 4111 \cdot 944731$.
 BH, 5689, 107941*, 31 : $31 \in X$.
 BH, 5689, 107941, 3 : $3 \in X$.

 BI, 5953 : $\Phi_7(5953) = 4663 \cdot 1352107 \cdot 1591927 \cdot 4434949$.
 BI, 5953, 1591927 : 1591927 is inadmissible.

 BJ, 10889 : $\Phi_7(10889) = 2003 \cdot 22093 \cdot 116341 \cdot 471997 \cdot 686057$.
 BJ, 10889, 471997 : $\Phi_2(471997) = 2 \cdot 19 \cdot 12421$.
 BJ, 10889, 471997*, 19 : $19 \in X$.

 BK, 17609 : $\Phi_7(17609) = 3109 \cdot 15289 \cdot 116747 \cdot 347873 \cdot 15444241$.
 BK, 17609, 116747 : 116747 is inadmissible.

 BL, 26833 : $\Phi_7(26833) = 26041 \cdot 11780777 \cdot 19965779 \cdot 60941581$.
 BL, 26833, 19965779 : 19965779 is inadmissible.

 BM, 48311 : $\Phi_7(48311) = 6751571 \cdot 7550173 \cdot 8327677 \cdot 29950187$.
 BM, 48311, 7550173 : $\Phi_2(7550173) = 2 \cdot 31 \cdot 47 \cdot 2591$.
 BM, 48311, 7550173*, 31 : $31 \in X$.

 BN, 56431 : $\Phi_7(56431) = 11047 \cdot 187573 \cdot 597367 \cdot 1912541 \cdot 13641041$.
 BN, 56431, 11047, 597367 : $\Phi_3(597367) = 3 \cdot 19 \cdot 73 \cdot 85760137$.
 BN, 56431, 11047, 597367, 3 : $3 \in X$.

 BO, 63587 : $\Phi_7(63587) = 13063 \cdot 380059 \cdot 1341257 \cdot 1632751 \cdot 6079823$.
 BO, 63587, 13063 : $\Phi_3(13063) = 3 \cdot 56885011$.
 BO, 63587, 13063, 3 : $3 \in X$.

 BP, 71209 : $\Phi_7(71209) = 46439 \cdot 128969 \cdot 160651 \cdot 10145059 \cdot 13357009$.
 BP, 71209, 160651 : 160651 is inadmissible.

 BQ, 109793 : $\Phi_7(109793) = 2731 \cdot 6287 \cdot 46831 \cdot 314581 \cdot 950111 \cdot 7288639$.
 BQ, 109793, 2731 : $\Phi_3(2731) = 3 \cdot 61 \cdot 40771$.
 BQ, 109793, 2731, 61 : $61 \in X$.

 BR, 113287 : $\Phi_7(113287) = 2339 \cdot 3319 \cdot 5419 \cdot 2085931 \cdot 4027927 \cdot 5980619$.

BR, 113287, 5419 : $\Phi_3(5419) = 3 \cdot 31 \cdot 313 \cdot 1009$.
BR, 113287, 5419, 3 : $3 \in X$.

BS, 140009 : $\Phi_7(140009) = 35869 \cdot 545651 \cdot 1282681 \cdot 9028013 \cdot 33234853$.
BS, 140009, 9028013 : $\Phi_2(9028013) = 2 \cdot 3 \cdot 1504669$.
BS, 140009, 9028013*, 3 : $3 \in X$.

BT, 187049 : $\Phi_7(187049) = 351779 \cdot 511463 \cdot 1506653 \cdot 5043011 \cdot 31329061$.
BT, 187049, 5043011 : 5043011 is inadmissible.

BU, 191693 : $\Phi_7(191693) = 7561 \cdot 11887 \cdot 14869 \cdot 16759 \cdot 89839 \cdot 118399 \cdot 208279$.
BU, 191693, 118399 : 118399 is inadmissible.

BV, 262351 : $\Phi_7(262351) = 17417 \cdot 304039 \cdot 7771457 \cdot 81400859 \cdot 97333853$.
BV, 262351, 81400859 : 81400859 is inadmissible.

BW, 266837 : $\Phi_7(266837) = 11173 \cdot 4856279 \cdot 8205037 \cdot 11347981 \cdot 71450513$.
BW, 266837, 4856279 : 4856279 is inadmissible.

BX, 312979 : $\Phi_7(312979) = 5419 \cdot 104987 \cdot 115123 \cdot 520241 \cdot 849703 \cdot 32464153$.
BX, 312979, 5419 : $\Phi_3(5419) = 3 \cdot 31 \cdot 313 \cdot 1009$.
BX, 312979, 5419, 3 : $3 \in X$.

BY, 339943 : $\Phi_7(339943) = 310577 \cdot 677447 \cdot 5376673 \cdot 14529761 \cdot 93890399$.
BY, 339943, 677447 : 677447 is inadmissible.

BZ, 402383 : $\Phi_7(402383) = 8093 \cdot 61657 \cdot 85597 \cdot 878221 \cdot 5240887 \cdot 21591347$.
BZ, 402383, 5240887 : $\Phi_3(5240887) = 3 \cdot 1153 \cdot 7867 \cdot 1009369$.
BZ, 402383, 5240887, 3 : $3 \in X$.

CA, 502339 : $\Phi_7(502339) = 5657 \cdot 42491 \cdot 46187 \cdot 4107881 \cdot 5256413 \cdot 67030433$.
CA, 502339, 42491 : $\Phi_3(42491) = 19 \cdot 139 \cdot 683653$.
CA, 502339, 42491, 19 : $19 \in X$.

CB, 594533 : $\Phi_7(594533) = 10613 \cdot 27917 \cdot 61643 \cdot 1356083 \cdot 30972047 \cdot 57571781$.
CB, 594533, 1356083 : 1356083 is inadmissible.

CC, 663823 : $\Phi_7(663823) = 2129 \cdot 8681 \cdot 2538733 \cdot 5864489 \cdot 11462851 \cdot 27128711$.
CC, 663823, 27128711 : 27128711 is inadmissible.

CD, 737981 : $\Phi_7(737981) = 1303 \cdot 107927 \cdot 845531 \cdot 1981267 \cdot 10305569 \cdot 66535379$.
CD, 737981, 107927 : 107927 is inadmissible.

CE, 746939 : $\Phi_7(746939) = 2129 \cdot 2927 \cdot 3851 \cdot 39901 \cdot 127079 \cdot 31370683 \cdot 45494401$.
CE, 746939, 127079 : 127079 is inadmissible.

CF, 755239 : $\Phi_7(755239) = 5153 \cdot 11887 \cdot 583493 \cdot 1096859 \cdot 61078081 \cdot 77500193$.
CF, 755239, 61078081 : $\Phi_2(61078081) = 2 \cdot 13 \cdot 113 \cdot 20789$.
CF, 755239, 61078081*, 13 : $13 \in X$.

CG, 820789 : $\Phi_7(820789) = 1877 \cdot 10039 \cdot 3678823 \cdot 7103699 \cdot 7727273 \cdot 80355437$.
CG, 820789, 3678823 : 3678823 is inadmissible.

CH, 1017209 : $\Phi_7(1017209) = 7057 \cdot 127289 \cdot 376853 \cdot 1339297 \cdot 46078537 \cdot 53027731$.
CH, 1017209, 46078537 : $\Phi_2(46078537) = 2 \cdot 11 \cdot 179 \cdot 11701$.
CH, 1017209, 46078537*, 11 : $11 \in X$.

CI. 1149521 : $\Phi_7(1149521) = 15373 \cdot 138181 \cdot 1419839 \cdot 1479059 \cdot 8363671 \cdot 61840549$.
 CI, 1149521, 1479059 : 1479059 is inadmissible.

CJ. 1732909 : $\Phi_7(1732909) = 69259 \cdot 1282471 \cdot 1483021 \cdot 2094107 \cdot 2241709 \cdot 43793093$.
 CJ, 1732909, 1282471 : 1282471 is inadmissible.

CK. 1742537 : $\Phi_7(1742537) = 1303 \cdot 7393 \cdot 22093 \cdot 36037 \cdot 5348449 \cdot 17815267 \cdot 38309251$.
 CK, 1742537, 17815267 : 17815267 is inadmissible.

CL. 1951819 : $\Phi_7(1951819) = 97007 \cdot 330611 \cdot 630169 \cdot 10342907 \cdot 11740093 \cdot 22529207$.
 CL, 1951819, 97007 : 97007 is inadmissible.

CM. 2110763 : $\Phi_7(2110763) = 13063 \cdot 378379 \cdot 2771693 \cdot 5800481 \cdot 25065013 \cdot 44400721$.
 CM, 2110763, 378379 : 378379 is inadmissible.

CN. 2299163 : $\Phi_7(2299163) = 38011 \cdot 259169 \cdot 659569 \cdot 5430923 \cdot 42765997 \cdot 97879993$.
 CN, 2299163, 5430923 : 5430923 is inadmissible.

CO. 3088091 : $\Phi_7(3088091) = 1499 \cdot 2801 \cdot 7547 \cdot 1135247 \cdot 24173101 \cdot 25799593 \cdot 38655919$.
 CO, 3088091, 1135247 : $\Phi_3(1135247) = 13 \cdot 229 \cdot 271 \cdot 1249 \cdot 1279$.
 CO, 3088091, 1135247, 13 : $13 \in X$.

CP. 4480403 : $\Phi_7(4480403) = 8821 \cdot 70099 \cdot 208993 \cdot 734021 \cdot 1252483 \cdot 1369607 \cdot 49712419$.
 CP, 4480403, 70099 : 70099 is inadmissible.

CQ. 5418277 : $\Phi_7(5418277) = 197023 \cdot 657959 \cdot 6573071 \cdot 28477093 \cdot 29668493 \cdot 35147309$.
 CQ, 5418277, 657959 : $\Phi_3(657959) = 13 \cdot 61 \cdot 271 \cdot 907 \cdot 2221$.
 CQ, 5418277, 657959, 13 : $13 \in X$.

CR. 5654071 : $\Phi_7(5654071) = 2927 \cdot 7547 \cdot 59333 \cdot 5776457 \cdot 9909593 \cdot 14802173 \cdot 29419237$.
 CR, 5654071, 5776457 : $\Phi_2(5776457) = 2 \cdot 3 \cdot 962743$.
 CR, 5654071, 5776457*, 3 : $3 \in X$.

CS. 6414623 : $\Phi_7(6414623) = 3067 \cdot 59011 \cdot 190093 \cdot 246289 \cdot 2459801 \cdot 36326963 \cdot 92011039$.
 CS, 6414623, 36326963 : 36326963 is inadmissible.

CT. 6450307 : $\Phi_7(6450307) = 7253 \cdot 32789 \cdot 33629 \cdot 46327 \cdot 251063 \cdot 331339 \cdot 901279 \cdot 2592829$.
 CT, 6450307, 901279 : 901279 is inadmissible.

CU. 6499631 : $\Phi_7(6499631) = 4663 \cdot 22247 \cdot 482441 \cdot 751871 \cdot 5595059 \cdot 18452113 \cdot 19406941$.
 CU, 6499631, 5595059 : 5595059 is inadmissible.

CV. 7163917 : $\Phi_7(7163917) = 13903 \cdot 232877 \cdot 460013 \cdot 490169 \cdot 2888747 \cdot 3732499 \cdot 17172877$.
 CV, 7163917, 2888747 : 2888747 is inadmissible.

CW. 7532381 : $\Phi_7(7532381) = 24179 \cdot 3786119 \cdot 15453887 \cdot 49118371 \cdot 49365779 \cdot 53241889$.
 CW, 7532381, 49118371 : 49118371 is inadmissible.

CX. 8104141 : $\Phi_7(8104141) = 19937 \cdot 34217 \cdot 729191 \cdot 1015561 \cdot 1699111 \cdot 5768869 \cdot 57211127$.
 CX, 8104141, 57211127 : 57211127 is inadmissible.

CY. 9997441 : $\Phi_7(9997441) = 1373 \cdot 20707 \cdot 1089383 \cdot 2475019 \cdot 8841449 \cdot 15675983 \cdot 93978403$.
 CY, 9997441, 1089383 : 1089383 is inadmissible.

CZ. 14257819 : $\Phi_7(14257819) = 69623 \cdot 148457 \cdot 244301 \cdot 1530019 \cdot 3633029 \cdot 7253107 \cdot 82518283$.
 CZ, 14257819, 69623 : $\Phi_3(69623) = 19 \cdot 37 \cdot 43 \cdot 160357$.
 CZ, 14257819, 69623, 19 : $19 \in X$.

DA. 16606193 : $\Phi_7(16606193) = 2591 \cdot 1339409 \cdot 1348747 \cdot 2516963 \cdot 2541701 \cdot 10134433 \cdot 69104869$.
DA, 16606193, 1348747 : 1348747 is inadmissible.

DB. 16955431 : $\Phi_7(16955431) = 3319 \cdot 815809 \cdot 1018907 \cdot 3669373 \cdot 10842119 \cdot 11932439 \cdot 18142097$.
DB, 16955431, 1018907 : 1018907 is inadmissible.

DC. 18786679 : $\Phi_7(18786679) = 6637 \cdot 852881 \cdot 1694393 \cdot 3354037 \cdot 5747239 \cdot 6603437 \cdot 36010451$.
DC, 18786679, 5747239 : 5747239 is inadmissible.

DD. 21101371 : $\Phi_7(21101371) = 1471 \cdot 25621 \cdot 29723 \cdot 38767 \cdot 126757 \cdot 15828457 \cdot 20260003 \cdot 50009261$.
DD, 21101371, 20260003 : 20260003 is inadmissible.

DE. 23150119 : $\Phi_7(23150119) = 50821 \cdot 638359 \cdot 1123403 \cdot 2347549 \cdot 4149881 \cdot 20643421 \cdot 21001177$.
DE, 23150119, 638359 : 638359 is inadmissible.

DF. 28234649 : $\Phi_7(28234649) = 1163 \cdot 8807 \cdot 161561 \cdot 500459 \cdot 1078757 \cdot 2027411 \cdot 6028457 \cdot 46399151$.
DF, 28234649, 2027411 : 2027411 is inadmissible.

DG. 28361629 : $\Phi_7(28361629) = 7477 \cdot 292531 \cdot 640949 \cdot 3785531 \cdot 29562317 \cdot 54942413 \cdot 60379747$.
DG, 28361629, 3785531 : 3785531 is inadmissible.

DH. 28578083 : $\Phi_7(28578083) = 1362551 \cdot 27985189 \cdot 46422433 \cdot 53190257 \cdot 66210901 \cdot 87383227$.
DH, 28578083, 1362551 : 1362551 is inadmissible.

DI. 30314399 : $\Phi_7(30314399) = 15443 \cdot 199697 \cdot 2437219 \cdot 2907577 \cdot 5152463 \cdot 69163459 \cdot 99649061$.
DI, 30314399, 2437219 : 2437219 is inadmissible.

DJ. 30823421 : $\Phi_7(30823421) = 2003 \cdot 6553 \cdot 25579 \cdot 249859 \cdot 2857709 \cdot 3456377 \cdot 14080837 \cdot 73505153$.
DJ, 30823421, 25579 : $\Phi_3(25579) = 3 \cdot 1129 \cdot 193183$.
DJ, 30823421, 25579, 3 : $3 \in X$.

DK. 33729187 : $\Phi_7(33729187) = 65003 \cdot 240899 \cdot 662369 \cdot 3182341 \cdot 21742757 \cdot 27410237 \cdot 74850161$.
DK, 33729187, 65003 : $\Phi_3(65003) = 13 \cdot 487 \cdot 667423$.
DK, 33729187, 65003, 13 : $13 \in X$.

DL. 39434663 : $\Phi_7(39434663) = 3557 \cdot 3613 \cdot 8387 \cdot 14281 \cdot 19181 \cdot 292223 \cdot 832721 \cdot 22835639 \cdot 22922047$.
DL, 39434663, 292223 : 292223 is inadmissible.

DM. 41981081 : $\Phi_7(41981081) = 3221 \cdot 19237 \cdot 140057 \cdot 503231 \cdot 2538803 \cdot 3414181 \cdot 8413021 \cdot 17189131$.
DM, 41981081, 2538803 : 2538803 is inadmissible.

DN. 44894383 : $\Phi_7(44894383) = 1933 \cdot 14869 \cdot 26209 \cdot 150893 \cdot 702913 \cdot 27240151 \cdot 41178523 \cdot 91355993$.
DN, 44894383, 27240151 : 27240151 is inadmissible.

DO. 50837341 : $\Phi_7(50837341) = 1163 \cdot 2843 \cdot 23899 \cdot 30059 \cdot 56393 \cdot 1055881 \cdot 3963317 \cdot 5527481 \cdot 5571343$.
DO, 50837341, 2843 : $\Phi_3(2843) = 13 \cdot 67 \cdot 9283$.
DO, 50837341, 2843, 13 : $13 \in X$.

DP. 61964431 : $\Phi_7(61964431) = 2521 \cdot 9829 \cdot 168869 \cdot 787879 \cdot 1712383 \cdot 2880739 \cdot 57176701 \cdot 60875039$.
DP, 61964431, 787879 : 787879 is inadmissible.

DQ. 68113807 : $\Phi_7(68113807) = 10613 \cdot 30829 \cdot 83791 \cdot 574813 \cdot 3701881 \cdot 10213001 \cdot 12842411 \cdot 13051697$.
DQ, 68113807, 12842411 : 12842411 is inadmissible.

DR. 74306809 : $\Phi_7(74306809) = 3851 \cdot 38921 \cdot 604031 \cdot 662719 \cdot 1496167 \cdot 3987229 \cdot 15379967 \cdot 30578689$.
DR, 74306809, 1496167 : 1496167 is inadmissible.

DS. 93058573 : $\Phi_7(93058573) = 10459 \cdot 16927 \cdot 399281 \cdot 512597 \cdot 1225183 \cdot 9568931 \cdot 35313391 \cdot 43292201$.

DS, 93058573, 1225183 : 1225183 is inadmissible.

DT, 96397919 : $\Phi_7(96397919) = 7001 \cdot 35533 \cdot 314693 \cdot 627901 \cdot 5457971 \cdot 5471887 \cdot 9429869 \cdot 57964453$.

DT, 96397919, 5471887 : 5471887 is inadmissible.

A.5 Table of acceptable values

p	r	$\Phi_r(p)$
3	7	1093
3	11	23 · 3851
3	13	797161
3	17	1871 · 34511
3	19	1597 · 363889
3	29	59 · 28537 · 20381027
3	31	683 · 102673 · 4404047
3	47	1223 · 21997 · 5112661 · 96656723
5	7	19531
5	11	12207031
5	19	191 · 6271 · 3981071
7	7	29 · 4733
7	11	1123 · 293459
11	7	43 · 45319
11	11	15797 · 1806113
13	7	5229043
13	11	23 · 419 · 859 · 18041
13	13	53 · 264031 · 1803647
17	7	25646167
19	7	701 · 70841
19	11	104281 · 62060021
23	7	29 · 5336717
29	7	7 · 88009573
31	13	42407 · 2426789 · 7908811
37	7	71 · 37140797
43	7	7 · 5839 · 158341
47	13	53 · 2237 · 14050609 · 71265169
59	7	43 · 281 · 757 · 4691
67	7	175897 · 522061
71	7	7 · 883 · 21020917
79	7	281 · 337 · 1289 · 2017
83	13	1249 · 1396513 · 1423319 · 43580447
97	7	43 · 967 · 20241187
109	7	113 · 281 · 53306107
113	7	7 · 44983 · 6670903
127	7	7 · 43 · 86353 · 162709
131	7	127 · 189967 · 211093
139	7	29 · 2857 · 87683177
167	11	23 · 89 · 331 · 397 · 1013 · 32099 · 1940599
173	7	3144079 · 8576317
191	7	127 · 197 · 10627 · 183569
191	13	131 · 1483 · 9049 · 92041 · 301627 · 48552947
197	7	7 · 29 · 97847 · 2957767

p	r	$\Phi_r(p)$
199	7	29 · 211 · 883 · 11552213
211	7	7 · 307189 · 41233879
223	7	29 · 491 · 1709 · 5076443
239	7	7 · 29 · 245561 · 3754507
269	7	43 · 211 · 631 · 2633 · 25229
271	7	9170197 · 43355341
293	7	43 · 2197609 · 6718489
347	7	39577763 · 44236319
359	7	211 · 449 · 1303 · 4019 · 4327
367	7	113 · 233437 · 92882357
389	7	127 · 337 · 659 · 827 · 148933
397	7	29 · 127 · 927137 · 1149457
397	11	11 · 23 · 67 · 3323 · 239273 · 344587 · 20993369
401	7	29 · 337 · 263047 · 1621397
409	7	6133 · 15919 · 48063373
431	7	29 · 953 · 967 · 1009 · 238267
439	7	7883 · 63841 · 14255627
509	7	29 · 2801 · 10333 · 20759803
601	7	631 · 4832521 · 15479857
607	7	54517 · 415759 · 2210419
619	7	3389 · 3732919 · 4453751
653	7	21757 · 706763 · 5049773
691	11	59951 · 133717 · 183041 · 455489 · 37187767
769	7	197 · 12342821 · 85161343
853	7	2647 · 11824121 · 12321989
887	7	5167 · 6271651 · 15045661
919	7	29 · 43 · 3851 · 21407 · 5866379
953	7	7 · 29 · 71 · 113 · 127 · 379 · 9566159
977	7	29 · 5573 · 914047 · 5893273
1063	7	337 · 2423 · 1289513 · 1371511
1123	7	113 · 3823 · 293147 · 15852481
1181	7	71 · 27791 · 202021 · 6812527
1279	7	56701 · 3745631 · 20627531
1283	7	29 · 631 · 3739 · 24781 · 2632673
1301	7	29 · 43 · 58735811 · 66256471
1303	7	7 · 67579 · 190261 · 54417721
1451	7	2381 · 52584967 · 74590391
1453	7	1051 · 11117 · 363119 · 2219491
1481	7	953 · 2087 · 265007 · 20033231
1523	7	71 · 337 · 449 · 235537 · 4935043
1531	7	29 · 631 · 9247939 · 76148717
1693	7	43 · 337 · 7673 · 37171 · 5700731
1823	7	29 · 71 · 547 · 5709019 · 5711539
1879	7	29 · 2017 · 20664827 · 36429751
1949	7	71 · 113 · 3137 · 206263 · 10563827

p	r	$\Phi_r(p)$
2003	7	$7 \cdot 1289 \cdot 10627 \cdot 33601 \cdot 20053433$
2053	7	$29 \cdot 161869 \cdot 3482179 \cdot 4582817$
2141	7	$29 \cdot 1163 \cdot 11719 \cdot 112967 \cdot 2158157$
2381	7	$7 \cdot 43 \cdot 2689 \cdot 3613 \cdot 72997 \cdot 853903$
2383	7	$475637 \cdot 6770429 \cdot 56889841$
2473	11	$23 \cdot 463 \cdot 2927 \cdot 8647 \cdot 81071 \cdot 451793 \cdot 640531 \cdot 1353551$
2503	7	$757 \cdot 19013 \cdot 3591869 \cdot 4758517$
2633	7	$7 \cdot 29 \cdot 232919 \cdot 2103613 \cdot 3351223$
2657	7	$71 \cdot 631 \cdot 1289 \cdot 1991389 \cdot 3060667$
2713	7	$29^2 \cdot 43 \cdot 73361 \cdot 258469 \cdot 581729$
3041	7	$29 \cdot 337 \cdot 9871 \cdot 811651 \cdot 10103759$
3083	7	$71 \cdot 743 \cdot 4481 \cdot 62189 \cdot 58431409$
3089	7	$1303 \cdot 89237 \cdot 316793 \cdot 23592997$
3221	7	$7 \cdot 673 \cdot 10333 \cdot 248879 \cdot 92204351$
3301	7	$29^2 \cdot 911 \cdot 38669 \cdot 186733 \cdot 233941$
3313	7	$29 \cdot 211 \cdot 216259 \cdot 884857 \cdot 1129619$
3623	7	$43^3 \cdot 35911 \cdot 353263 \cdot 2242843$
3779	7	$197 \cdot 2311 \cdot 23773 \cdot 455407 \cdot 591053$
4013	7	$71 \cdot 281 \cdot 9829 \cdot 438047 \cdot 48632711$
4129	7	$5867 \cdot 17053 \cdot 714463 \cdot 69339047$
4133	7	$547 \cdot 54083 \cdot 5997223 \cdot 28099541$
4241	7	$29 \cdot 197 \cdot 137957 \cdot 463303 \cdot 15938189$
4243	7	$7 \cdot 421 \cdot 953 \cdot 8443 \cdot 9157 \cdot 26879273$
4283	7	$29 \cdot 29611 \cdot 41539 \cdot 100003 \cdot 1730891$
4289	7	$1471 \cdot 8807 \cdot 9619 \cdot 32467 \cdot 1538951$
4327	7	$7 \cdot 3221 \cdot 5503 \cdot 5657 \cdot 92401 \cdot 101221$
4423	7	$113 \cdot 180797 \cdot 5720863 \cdot 64072051$
4621	7	$7 \cdot 2532587 \cdot 22078253 \cdot 24881851$
4801	7	$29 \cdot 1583 \cdot 6959 \cdot 3578737 \cdot 10713347$
4957	7	$7 \cdot 127 \cdot 449 \cdot 8387 \cdot 68279 \cdot 64917119$
5399	7	$2731 \cdot 9941 \cdot 14811889 \cdot 61602479$
5689	7	$9479 \cdot 107941 \cdot 338773 \cdot 97821473$
5953	7	$4663 \cdot 1352107 \cdot 1591927 \cdot 4434949$
6067	7	$43 \cdot 71 \cdot 1373 \cdot 6581 \cdot 32803 \cdot 55120451$
6577	7	$29 \cdot 43 \cdot 71 \cdot 40993 \cdot 644029 \cdot 34633243$
6917	7	$7 \cdot 71 \cdot 187909 \cdot 13749289 \cdot 85307111$
7027	7	$491 \cdot 298327 \cdot 25883593 \cdot 31760177$
7759	7	$29^2 \cdot 71 \cdot 337 \cdot 1051 \cdot 2213 \cdot 2689 \cdot 1733929$
8009	7	$7 \cdot 43 \cdot 127 \cdot 491 \cdot 127247 \cdot 305873 \cdot 361313$
8053	7	$29 \cdot 71 \cdot 113 \cdot 379 \cdot 673 \cdot 67537 \cdot 68056451$
8243	7	$29 \cdot 883 \cdot 13259 \cdot 9842603 \cdot 93882587$
8681	7	$7 \cdot 62903 \cdot 285979 \cdot 1563101 \cdot 2174593$
9103	7	$631 \cdot 911 \cdot 514417 \cdot 1338331 \cdot 1437899$
9397	7	$617 \cdot 8779 \cdot 11579 \cdot 1551383 \cdot 7077197$
9403	7	$29 \cdot 2311 \cdot 2633 \cdot 9521 \cdot 54629 \cdot 7531651$

p	r	$\Phi_r(p)$
9463	7	71 · 6791 · 15667 · 4172099 · 22786961
9539	7	673 · 3504929 · 6483611 · 49266463
9719	7	281 · 3067 · 8219 · 19937 · 30773 · 193957
9787	7	7 · 127 · 1385441 · 7148849 · 99819637
9967	7	29 · 6427 · 47293 · 8157031 · 13636253
10889	7	2003 · 22093 · 116341 · 471997 · 686057
10939	7	197 · 1093 · 205913 · 2212673 · 17466989
10949	7	7 · 29 · 197 · 547 · 1009 · 6917 · 25523 · 442177
11549	7	29 · 449 · 236909 · 21238211 · 36220969
11987	7	211 · 463 · 673 · 2689 · 314161 · 53416777
12253	7	43 · 113 · 379 · 113723 · 257993 · 62639417
12553	7	29 · 701 · 1471 · 49463 · 594119 · 4452841
12637	7	127 · 701 · 641747 · 732299 · 97347377
12917	7	953 · 6007 · 352661 · 442961 · 5194337
13513	7	43 · 1303 · 122081 · 23736637 · 37502711
13723	7	71 · 239 · 1492163 · 4937941 · 53420459
15307	7	29 · 883 · 2381 · 51437 · 1001953 · 4093811
15313	7	71 · 463 · 547 · 44633 · 3918587 · 4099943
15601	7	43 · 2423 · 3335263 · 5823749 · 7125049
17011	7	7 · 6217 · 311963 · 18293101 · 97574261
17431	7	7 · 43 · 127 · 3347 · 50527 · 558881 · 7764079
17477	7	547 · 2549 · 13931 · 26619181 · 55117819
17491	7	239 · 827 · 1093 · 12097 · 2287951 · 4789219
17609	7	3109 · 15289 · 116747 · 347873 · 15444241
18397	7	7 · 1933 · 4957 · 32411 · 1343917 · 13270643
18859	7	7 · 5237 · 6833 · 39019 · 184913 · 24894269
19441	7	421 · 2143 · 4663 · 15233 · 383489 · 2196979
20441	7	7 · 29 · 43 · 659 · 2213 · 27763 · 68279 · 3023077
21803	7	29^2 · 3221 · 128857 · 13311383 · 23120623
21961	7	547 · 171473 · 178613 · 640109 · 10461221
23339	7	7 · 29 · 1051 · 21841 · 92779 · 353137 · 1058639
23671	7	29 · 43 · 239 · 9829 · 90203 · 237301 · 2805587
24439	7	337 · 1723 · 6287 · 7043 · 576689 · 14370119
24683	7	7 · 113 · 2213 · 2843 · 18803 · 67033 · 36054103
24809	7	7 · 43 · 71 · 1303 · 362069 · 1266371 · 18262133
25237	7	29 · 71 · 9689 · 641327 · 1970417 · 10248743
25717	7	29 · 617 · 1667 · 305971 · 332851 · 95233517
26833	7	26041 · 11780777 · 19965779 · 60941581
27457	7	29 · 42463 · 65171 · 71261 · 91813 · 816047
27763	7	7 · 16073 · 7921691 · 10907849 · 47104457
30557	7	29 · 449 · 631 · 10067 · 90679 · 99667 · 1089047
31667	7	757 · 7309 · 1894873 · 3503711 · 27453203
32051	7	43 · 71 · 24151 · 488797 · 2163883 · 13900769
32653	7	43 · 113 · 827 · 831713 · 6545477 · 55409551
33409	7	43 · 127 · 10739 · 303731 · 2156183 · 36206213

p	r	$\Phi_r(p)$
35023	7	29 · 43 · 71 · 113 · 1499 · 4243 · 330877 · 87657277
35983	7	29 · 43 · 113 · 21001 · 30983 · 2938307 · 8057323
39607	7	7 · 43 · 421 · 2549 · 28477 · 18879253 · 22230293
41203	7	7 · 29 · 211 · 1163 · 6329 · 16339 · 340397 · 2790481
41759	7	617 · 48371 · 84463 · 25517689 · 82439029
41941	7	29 · 43 · 4649 · 1506457 · 13157341 · 47368777
42403	7	113 · 239 · 3851 · 140869 · 8153447 · 48661243
42577	7	71 · 127 · 827 · 2297 · 13567 · 2320361 · 11048227
43721	7	911 · 71317 · 1158823 · 6418259 · 14454553
45137	7	7 · 6091 · 47951 · 156269 · 428807 · 61728647
45263	7	7 · 29 · 225499 · 1291991 · 3314893 · 43863163
45341	7	113 · 127 · 13259 · 2515003 · 2804117 · 6474833
46327	7	7 · 43 · 127373 · 1260673 · 7805981 · 26202373
48131	7	29 · 113 · 213613 · 235747 · 3276491 · 22993181
48311	7	6751571 · 7550173 · 8327677 · 29950187
48473	7	113 · 30829 · 42743 · 241249 · 352073 · 1025669
48809	7	43 · 71 · 113 · 4831 · 97609 · 6382097 · 13023053
50261	7	7 · 5923 · 88747 · 168631 · 2245489 · 11570539
52631	7	29 · 71 · 38711 · 459383 · 7327601 · 79219211
53831	7	7 · 29 · 39341 · 104651 · 257489 · 269221 · 420001
55787	7	29 · 43 · 197 · 239 · 2292053 · 10808939 · 20723711
56053	7	29 · 743 · 104707 · 1805833 · 2248681 · 3385579
56431	7	11047 · 187573 · 597367 · 1912541 · 13641041
58453	7	43 · 71 · 2143 · 4019 · 4229 · 5264099 · 68142733
59467	7	43 · 1373 · 140449 · 469631 · 2847139 · 3988783
61909	7	7 · 29 · 4663 · 34693 · 130439 · 143333 · 91700729
63587	7	13063 · 380059 · 1341257 · 1632751 · 6079823
63659	7	7 · 127 · 211 · 827 · 414331 · 18362807 · 56388361
64661	7	29 · 13931 · 67061 · 233549 · 572909 · 20162633
64853	7	71 · 701 · 12433 · 800143 · 6421157 · 23402051
66529	7	7 · 9521 · 36107 · 50821 · 20364457 · 34816447
67021	7	337 · 21841 · 28183 · 183709 · 209189 · 11368757
67477	7	29 · 113 · 827 · 5237 · 27077 · 4379033 · 56091589
70223	7	43 · 211 · 1429 · 3978899 · 35870297 · 64803943
70913	7	743 · 5923 · 13693 · 119533 · 365513 · 48299371
71209	7	46439 · 128969 · 160651 · 10145059 · 13357009
73589	7	29 · 43 · 967 · 7757 · 52529 · 13546121 · 23860663
73819	7	113 · 127 · 3347 · 771653 · 65887067 · 66260503
74177	7	29 · 113 · 7043 · 9045947 · 9943207 · 80242933
75367	7	29 · 113 · 281 · 102593 · 159503 · 275059 · 44217881
75557	7	281 · 701 · 55217 · 168533 · 5049521 · 20101187
76753	7	43 · 379 · 1093 · 4995299 · 26776331 · 85809907
78653	7	7 · 2549 · 109481 · 1536221 · 4705681 · 16765421
80677	7	127 · 197 · 491 · 743 · 1051 · 24809 · 66403 · 17448733
83071	7	113 · 211 · 1093 · 1163 · 7393 · 23920163 · 61313519

p	r	$\Phi_r(p)$
85597	7	$7 \cdot 617 \cdot 39509 \cdot 8658679 \cdot 8851837 \cdot 30074059$
89083	7	$7 \cdot 29^2 \cdot 534199 \cdot 1846909 \cdot 7210043 \cdot 11934203$
90149	7	$43 \cdot 2143 \cdot 71429 \cdot 1223587 \cdot 2792189 \cdot 23868517$
91153	7	$281^2 \cdot 162359 \cdot 1860517 \cdot 2205449 \cdot 10904629$
92233	7	$7 \cdot 43 \cdot 1289 \cdot 1597 \cdot 6833 \cdot 25579 \cdot 513367 \cdot 11073259$
92503	7	$239 \cdot 247031 \cdot 8538797 \cdot 22452151 \cdot 55352291$
93251	7	$29 \cdot 2129 \cdot 2437 \cdot 25439 \cdot 148471 \cdot 385939 \cdot 2998031$
94253	7	$421 \cdot 1706363 \cdot 1886347 \cdot 22400533 \cdot 23096291$
96907	7	$211 \cdot 774691 \cdot 3782773 \cdot 31910201 \cdot 41974409$
99551	7	$29 \cdot 547 \cdot 673 \cdot 8387 \cdot 783931 \cdot 1180901 \cdot 11743019$
99623	7	$43 \cdot 119869 \cdot 48536153 \cdot 49730507 \cdot 78577549$
103981	7	$29 \cdot 71 \cdot 2129 \cdot 2339 \cdot 476029 \cdot 4352279 \cdot 59499833$
108949	7	$7 \cdot 29 \cdot 2549 \cdot 8779 \cdot 415031 \cdot 23933771 \cdot 37063027$
109793	7	$2731 \cdot 6287 \cdot 46831 \cdot 314581 \cdot 950111 \cdot 7288639$
113287	7	$2339 \cdot 3319 \cdot 5419 \cdot 2085931 \cdot 4027927 \cdot 5980619$
115337	7	$43 \cdot 659 \cdot 49043 \cdot 3420691 \cdot 5744957 \cdot 86195411$
118189	7	$7 \cdot 71 \cdot 57373 \cdot 60397 \cdot 230567 \cdot 1121051 \cdot 6122999$
120349	7	$43 \cdot 547 \cdot 358541 \cdot 1970291 \cdot 10184959 \cdot 17954539$
120569	7	$7 \cdot 29 \cdot 554233 \cdot 7019153 \cdot 58047053 \cdot 67013801$
120779	7	$7 \cdot 29 \cdot 43 \cdot 127 \cdot 11299 \cdot 3491881 \cdot 5076821 \cdot 13979533$
123493	7	$211 \cdot 449 \cdot 827 \cdot 57667 \cdot 143263 \cdot 880909 \cdot 6220579$
124601	7	$7 \cdot 757 \cdot 953 \cdot 5279 \cdot 455687 \cdot 10627373 \cdot 28986889$
124633	7	$29 \cdot 2801 \cdot 300931 \cdot 394577 \cdot 17641961 \cdot 22026481$
126551	7	$29 \cdot 2511167 \cdot 29664223 \cdot 38610461 \cdot 49247633$
128833	7	$211 \cdot 701 \cdot 1723 \cdot 160441 \cdot 161267 \cdot 205423 \cdot 3375751$
137699	7	$29 \cdot 71 \cdot 591319 \cdot 5263679 \cdot 24672733 \cdot 43112483$
140009	7	$35869 \cdot 545651 \cdot 1282681 \cdot 9028013 \cdot 33234853$
143729	7	$113 \cdot 9199 \cdot 89657 \cdot 1985047 \cdot 4602109 \cdot 10354723$
144407	7	$29 \cdot 113 \cdot 127 \cdot 2017 \cdot 9654247 \cdot 14866013 \cdot 75272009$
144577	7	$43 \cdot 491 \cdot 300301 \cdot 5468891 \cdot 14520353 \cdot 18139073$
146801	7	$197 \cdot 911 \cdot 2801 \cdot 49547 \cdot 343897 \cdot 403957 \cdot 2892667$
148549	7	$281 \cdot 1429 \cdot 1709 \cdot 44381 \cdot 62539 \cdot 91967 \cdot 61342387$
153107	7	$29 \cdot 3109 \cdot 12097 \cdot 16489523 \cdot 17273131 \cdot 41466517$
154183	7	$7 \cdot 449 \cdot 2017 \cdot 53719 \cdot 71387 \cdot 9396731 \cdot 58809521$
155833	7	$29 \cdot 127 \cdot 719027 \cdot 12534761 \cdot 14455757 \cdot 29843899$
156703	7	$7 \cdot 29 \cdot 43 \cdot 827 \cdot 5209 \cdot 4776647 \cdot 8992691 \cdot 9167047$
156887	7	$71 \cdot 6679 \cdot 27539 \cdot 1839923 \cdot 18567557 \cdot 33423517$
162209	7	$71 \cdot 617 \cdot 112687 \cdot 665141 \cdot 67552997 \cdot 82125247$
162973	7	$127 \cdot 1303 \cdot 47657 \cdot 3932993 \cdot 8453509 \cdot 71459767$
166207	7	$827 \cdot 4019 \cdot 11719 \cdot 3595733 \cdot 4190999 \cdot 35915293$
172489	7	$43 \cdot 71 \cdot 197 \cdot 1723 \cdot 8233 \cdot 231547 \cdot 2129107 \cdot 6261781$
174289	7	$631 \cdot 1307923 \cdot 20800529 \cdot 29855141 \cdot 54691183$
174637	7	$7 \cdot 71 \cdot 491 \cdot 3571 \cdot 33587 \cdot 315281 \cdot 832987 \cdot 3690499$
174749	7	$7 \cdot 29 \cdot 1933 \cdot 208699 \cdot 1906871 \cdot 2171261 \cdot 83986421$
181739	7	$29 \cdot 43^2 \cdot 127 \cdot 757 \cdot 4691 \cdot 44507 \cdot 3790739 \cdot 8831593$

p	r	$\Phi_r(p)$
187049	7	351779 · 511463 · 1506653 · 5043011 · 31329061
191693	7	7561 · 11887 · 14869 · 16759 · 89839 · 118399 · 208279
191801	7	7 · 29 · 43 · 71 · 11789 · 1607327 · 44459479 · 95354029
196657	7	449 · 617 · 140729 · 1982191 · 20355287 · 36772303
204599	7	71 · 127 · 883 · 5153 · 5419 · 81439 · 150907 · 26845981
218971	7	197 · 1471 · 2647 · 9871 · 1557613 · 2178877 · 4289783
219031	7	7 · 29 · 197 · 22541 · 174721 · 183317 · 771653 · 4955987
220217	7	29 · 211 · 239 · 2423 · 4201 · 41231 · 3201619 · 58040221
236641	7	449 · 23899 · 154981 · 735953 · 6614273 · 21692273
242617	7	43 · 7253 · 140603 · 3349501 · 16050539 · 86512469
244457	7	29 · 883 · 1583 · 3361 · 6203 · 32341 · 2406307 · 3244907
246319	7	71 · 1303 · 1709 · 6217 · 6553 · 87683 · 141121 · 2802311
254377	7	953 · 2269 · 882967 · 1007609 · 3497089 · 40271533
262351	7	17417 · 304039 · 7771457 · 81400859 · 97333853
264139	7	7 · 29 · 239 · 33811 · 49547 · 82601 · 620159 · 81571631
266837	7	11173 · 4856279 · 8205037 · 11347981 · 71450513
267139	7	29 · 71 · 7603 · 57793 · 814493 · 17651341 · 27941117
275719	7	29 · 127 · 421 · 12923 · 15359 · 25733 · 2735377 · 20280751
285559	7	7 · 29 · 449 · 1667 · 3389 · 14771 · 210071 · 224267 · 1513163
288109	7	29 · 6343 · 6553 · 17207 · 75181 · 6354013 · 57722911
290531	7	463 · 967 · 812491 · 819253 · 38938117 · 51824767
304501	7	7 · 883 · 69427 · 430739 · 532099 · 882967 · 9179003
311569	7	113 · 33391 · 1011599 · 2509963 · 4149811 · 23009911
312551	7	7 · 113 · 127 · 281 · 119183 · 1936747 · 3272473 · 43719677
312979	7	5419 · 104987 · 115123 · 520241 · 849703 · 32464153
313289	7	71 · 883 · 710627 · 6754567 · 55483079 · 56630897
317729	7	239 · 25873 · 60383 · 628391 · 57408821 · 76379101
327443	7	43 · 1429 · 9437 · 112603 · 156913 · 8109557 · 14834569
336373	7	239 · 673 · 3613 · 8821 · 120919 · 38839123 · 60167969
339943	7	310577 · 677447 · 5376673 · 14529761 · 93890399
344759	7	29 · 113 · 337 · 9157 · 25537 · 186761 · 776077 · 44861573
345311	7	7 · 43 · 491 · 617 · 2731 · 5279 · 53173 · 1412461 · 17170763
354551	7	7 · 43 · 71 · 6581 · 206501 · 386989 · 4076003 · 43361221
358223	7	239 · 8933 · 28057 · 13193419 · 34867841 · 76683713
392087	7	29 · 1303 · 11159 · 74201 · 895007 · 4153759 · 31235653
400837	7	211 · 127583 · 172243 · 1970921 · 5289901 · 85797433
401593	7	43 · 1289 · 14827 · 51871 · 153371 · 20750843 · 30920009
402383	7	8093 · 61657 · 85597 · 878221 · 5240887 · 21591347
404267	7	29 · 631 · 31627 · 156913 · 375103 · 2655577 · 48256643
406631	7	7 · 39971 · 926983 · 12504647 · 18117401 · 76934621
413477	7	7 · 29 · 1303 · 3147649 · 12927349 · 15906563 · 29187481
432923	7	7 · 197 · 1373 · 1429 · 6203 · 7211 · 57709 · 76231 · 12365893
434029	7	7 · 113 · 281 · 4523 · 16927 · 2966083 · 8201663 · 16148749
441841	7	7 · 43 · 127 · 281 · 449 · 9059 · 2587943 · 7274807 · 9045191
445229	7	7 · 337 · 463 · 1933 · 11579 · 210869 · 32508197 · 46481933

p	r	$\Phi_r(p)$
449333	7	29 · 71 · 1709 · 3137 · 4999 · 8681 · 12713 · 26993 · 50066339
460181	7	7 · 71 · 181609 · 222461 · 3397549 · 9759121 · 14264251
463987	7	29 · 224309 · 384889 · 4913497 · 9379861 · 86470609
469237	7	43 · 491 · 1051 · 7477 · 9227 · 205703 · 1287217 · 26334281
472457	7	43 · 1460957 · 2076803 · 2178653 · 4680733 · 8359331
472721	7	281 · 18523 · 41959 · 9156701 · 69643771 · 80124661
483503	7	43 · 6833 · 42071 · 93941 · 235607 · 640669 · 72888019
493397	7	29 ² · 127 · 1163 · 2129 · 4229 · 26041 · 50177 · 71359 · 138349
494677	7	7 · 29 · 71 · 491 · 1597 · 14533 · 595351 · 6774503 · 22119973
502339	7	5657 · 42491 · 46187 · 4107881 · 5256413 · 67030433
524389	7	43 · 3739 · 4397 · 203897 · 210533 · 9011059 · 76038761
529421	7	71 · 113 · 911 · 16087 · 45403 · 136319 · 1756567 · 17225503
534883	7	29 · 337 · 463 · 2900437 · 4130309 · 12971197 · 33305707
560093	7	29 · 71 · 109313 · 817279 · 1927507 · 6269971 · 13886783
563503	7	127 · 1933 · 2491847 · 14002283 · 50976143 · 73325561
575557	7	29 · 63463 · 102607 · 121633 · 188189 · 336827 · 24967937
578131	7	7 · 29 · 2633 · 159223 · 177269 · 188147 · 539323 · 24390829
578407	7	281 · 757 · 51871 · 80473 · 894181 · 953093 · 49483939
592919	7	43 · 6833 · 60607 · 237707 · 704243 · 2362571 · 6169087
593083	7	7 · 71 ² · 827 · 4481 · 6917 · 83791 · 15279601 · 37581251
594533	7	10613 · 27917 · 61643 · 1356083 · 30972047 · 57571781
595877	7	71 · 113 · 66809 · 136403 · 2231881 · 4590251 · 59763677
654343	7	29 · 1093 · 58451 · 79423 · 878641 · 21934291 · 27678463
655351	7	281 · 808039 · 1383509 · 3056789 · 4971583 · 16594481
663823	7	2129 · 8681 · 2538733 · 5864489 · 11462851 · 27128711
677021	7	29 · 690887 · 743989 · 7240073 · 23207297 · 38448061
692353	7	29 · 71 · 449 · 911 · 1499 · 128969 · 204331 · 1443989 · 2292767
704731	7	43 · 127 · 337 · 150011 · 304739 · 323009 · 626011 · 7200971
732157	7	29 · 113 · 127 · 239 · 2689 · 170213 · 341993 · 400331 · 24713137
737129	7	7 · 29 · 113 · 491 · 883 · 911 · 16073 · 109201 · 2807239 · 3593549
737981	7	1303 · 107927 · 845531 · 1981267 · 10305569 · 66535379
741809	7	43 · 127 · 197 · 211 · 1163 · 13903 · 200383 · 12055723 · 18792733
746939	7	2129 · 2927 · 3851 · 39901 · 127079 · 31370683 · 45494401
755239	7	5153 · 11887 · 583493 · 1096859 · 61078081 · 77500193
757271	7	29 · 43 · 16381 · 197009 · 16811131 · 36164437 · 77079157
784457	7	29 ² · 743 · 1429 · 18061 · 21997487 · 25506643 · 25753141
820789	7	1877 · 10039 · 3678823 · 7103699 · 7727273 · 80355437
822989	7	827 · 77491 · 1248353 · 5998217 · 14279539 · 45345497
844957	7	7 · 37493 · 150431 · 662551 · 889687 · 1603267 · 9753493
851297	7	127 · 211 · 4271 · 10151 · 52963 · 804329 · 1531181 · 5022613
882461	7	29 · 449 · 186271 · 231631 · 7643609 · 9319619 · 11800237
899291	7	7 · 127 · 337 · 3319 · 8821 · 11159 · 282703 · 330569 · 57826567
921259	7	29 · 178151 · 9267917 · 16071007 · 27967367 · 28407163
923599	7	29 · 757 · 3823 · 16493 · 223217 · 422689 · 1165711 · 4077221
930481	7	29 · 43 · 1429 · 4789 · 14281 · 14827 · 35099 · 542123 · 18875459

p	r	$\Phi_r(p)$
971039	7	113 · 491 · 71527 · 2404823 · 2447551 · 4459687 · 8047691
982339	7	7 · 43 · 113 · 1597 · 4733 · 6637 · 5035339 · 8371063 · 12493993
982603	7	29 · 127 · 197 · 883 · 1933 · 2437 · 44059 · 78041321 · 86734859
998633	7	127 · 70841 · 1564907 · 15453299 · 56513297 · 80665369
1006933	7	29 · 9283 · 65843 · 93703 · 299419 · 22153909 · 94607311
1017209	7	7057 · 127289 · 376853 · 1339297 · 46078537 · 53027731
1019663	7	7 · 29 · 43 · 71 · 8513 · 288527 · 1269311 · 5819773 · 99948059
1027931	7	43 · 2927 · 24876671 · 59459527 · 64857563 · 97705427
1028329	7	7 · 421 · 49057 · 630827 · 22137277 · 22320607 · 26240453
1054373	7	29 · 547 · 883 · 3221 · 16871 · 263201 · 74239831 · 92377027
1132721	7	127 · 239 · 337 · 1289 · 246289 · 842003 · 12481771 · 61889759
1135451	7	239 · 827 · 4663 · 42967 · 89041 · 223007 · 309583 · 8802809
1149521	7	15373 · 138181 · 1419839 · 1479059 · 8363671 · 61840549
1149803	7	113 · 547 · 2143 · 4691 · 29723 · 46187 · 39756991 · 68133661
1233187	7	29 · 197 · 373003 · 887153 · 3892631 · 5162011 · 92584171
1236757	7	29 · 135017 · 176849 · 356077 · 379849 · 882491 · 43296457
1242781	7	7 · 547 · 13903 · 48413 · 265511 · 1095403 · 2171261 · 2263829
1246631	7	7 · 4397 · 458963 · 647809 · 1381997 · 16704157 · 17767121
1249099	7	43 · 47251 · 119687 · 277747 · 524189 · 1847539 · 58066303
1286953	7	29 · 113 · 25999 · 6940823 · 14275031 · 16237369 · 33146653
1310251	7	43 · 113 · 3347 · 25733 · 855191 · 1249669 · 1599809 · 7071443
1323967	7	7 · 71 · 12097 · 17389 · 446041 · 975157 · 1999957 · 59222213
1328923	7	7 · 421 · 107269 · 229139 · 10740689 · 80014481 · 88480001
1385441	7	7 · 29 · 13693 · 342203 · 353081 · 905213 · 934613 · 24888179
1416931	7	29 · 1429 · 39971 · 75181 · 24696421 · 30183203 · 87178589
1417831	7	43 · 71 · 302597 · 5249539 · 7665491 · 8279489 · 26393137
1458727	7	71 · 5531 · 31991 · 135787 · 6083141 · 30098699 · 30847559
1484701	7	7 · 71 · 39313 · 5420017 · 26778473 · 45707803 · 82635169
1485019	7	29 · 127 · 421 ² · 449 · 283669 · 769231 · 2228927 · 75234251
1486847	7	113 · 27749 · 46957 · 180503 · 419651 · 22780171 · 42524791
1545217	7	127 · 1499 · 78583 · 1595273 · 4021249 · 4059581 · 34940249
1630021	7	7 · 113 · 13679 · 477863 · 513899 · 984859 · 1458619 · 4913959
1652921	7	127 · 281 · 421 · 110083 · 140869 · 1516369 · 3468851 · 16641577
1660063	7	29 · 32117 · 101081 · 582541 · 2594677 · 6035737 · 24367169
1709483	7	29 ² · 197 · 953 · 24809 · 305621 · 358667 · 6433841 · 9033991
1732909	7	69259 · 1282471 · 1483021 · 2094107 · 2241709 · 43793093
1740199	7	29 · 449 · 6917 · 7001 · 11383 · 39901 · 61223 · 337751 · 4689427
1740589	7	43 · 911 · 2969 · 4663 · 23059 · 5441311 · 7725271 · 52900219
1742537	7	1303 · 7393 · 22093 · 36037 · 5348449 · 17815267 · 38309251
1793101	7	673 · 953 · 147211 · 231967 · 6094229 · 7934921 · 31382891
1795247	7	113 · 8779 · 305761 · 418181 · 683957 · 18912461 · 20403223
1827311	7	281 · 967 · 869233 · 1005551 · 1834981 · 2774129 · 30792413
1899047	7	127 · 197 · 337 · 164837 · 908573 · 1938161 · 2651111 · 7229069
1909489	7	7 · 617 · 5531 · 42043 · 91939 · 4324979 · 4671101 · 25984813
1918537	7	743 · 28813 · 137453 · 340201 · 359549 · 1970543 · 70308883

p	r	$\Phi_r(p)$
1939331	7	71 · 2689 · 4523 · 17683 · 33181 · 219647 · 12133031 · 39399991
1950853	7	29 ² · 1877 · 3823 · 7127 · 42169 · 572881 · 3453563 · 15362117
1951819	7	97007 · 330611 · 630169 · 10342907 · 11740093 · 22529207
1992937	7	43 · 10753 · 101627 · 678959 · 1224217 · 31710071 · 50588903
2090681	7	43 · 1583 · 2521 · 12391 · 25229 · 33349 · 97259 · 122921 · 3904447
2091247	7	911 · 1481971 · 4069801 · 10024939 · 33620959 · 45165737
2099711	7	29 · 43 · 281 · 1031423 · 2022469 · 2554819 · 3263639 · 14060593
2110763	7	13063 · 378379 · 2771693 · 5800481 · 25065013 · 44400721
2112419	7	7 · 2003 · 44647 · 6216841 · 21762931 · 25224053 · 41591761
2113399	7	7 · 29 · 43 · 35533 · 97021 · 148793 · 189169 · 1248101 · 84283949
2116801	7	7 · 197 · 379 · 617 · 631 · 8597 · 20749 · 30059 · 6434429 · 12815447
2125451	7	43 · 239 · 673 · 701 · 371029 · 13882247 · 60632573 · 60888283
2133811	7	7 · 29 · 71 · 197 · 6301 · 14533 · 56701 · 214607 · 344177 · 86683031
2160997	7	197 · 883 · 31249 · 5364731 · 8724619 · 12604243 · 31757839
2223443	7	449 · 3347 · 306139 · 1279937 · 1441133 · 1894411 · 75156859
2289697	7	281 · 919969 · 8282401 · 13419673 · 56015989 · 89532059
2299163	7	38011 · 259169 · 659569 · 5430923 · 42765997 · 97879993
2371309	7	463 · 3067 · 356287 · 615679 · 3672677 · 11356591 · 13685141
2467447	7	43 · 673 · 1667 · 16759 · 434113 · 4654609 · 11083759 · 12463697
2498939	7	197 · 491 · 23297 · 9553153 · 15389977 · 17070803 · 43057393
2522447	7	71 · 3851 · 14939 · 4860829 · 9970423 · 14705279 · 88487491
2522543	7	29 · 4957 · 7351 · 168491 · 168869 · 178613 · 6155129 · 7794557
2550857	7	7 · 43 · 239 · 827 · 1009 · 2322797 · 3777173 · 7615441 · 68688271
2602673	7	211 · 1709 · 117671 · 2200843 · 3193261 · 13923617 · 74859737
2613979	7	491 · 104651 · 114031 · 259547 · 2558683 · 2881621 · 28450871
2708753	7	43 · 71 · 113 · 547 · 22751 · 393191 · 624401 · 7833659 · 47840059
2838217	7	29 · 113 · 30059 · 145643 · 273043 · 1756021 · 1963081 · 38710981
2900297	7	7 · 29 · 1009 · 6917 · 37507 · 207061 · 716143 · 1198513 · 63022681
3051487	7	29 · 43 · 19993 · 125441 · 581743 · 2755369 · 8442337 · 19077073
3056393	7	29 · 211 · 547 · 2689 · 10487 · 15331 · 212479 · 33385129 · 79415617
3088091	7	1499 · 2801 · 7547 · 1135247 · 24173101 · 25799593 · 38655919
3124843	7	7 · 127 · 211 · 438887 · 2412341 · 4046953 · 28982843 · 39969119
3233441	7	7 · 71 · 239 · 2017 · 2731 · 3067 · 7841 · 891661 · 3063257 · 26591293
3239963	7	29 · 71 · 211 · 5209 · 4156643 · 30066457 · 45559193 · 89773111
3246073	7	29 · 197 · 1395997 · 1812959 · 37535891 · 37753829 · 57095963
3386909	7	7 · 211 · 421 · 803461 · 2042237 · 4908107 · 13329653 · 22612829
3447673	7	127 · 211 · 239 · 6287 · 6680521 · 9218581 · 14486221 · 46751923
3455317	7	29 · 967 · 1163 · 28631 · 352633 · 6653921 · 26951597 · 28820513
3481007	7	127 · 449 · 7211 · 41357 · 173573 · 269543 · 37372021 · 59838143
3487303	7	7 · 29 · 13721 · 516643 · 1216559 · 4960187 · 11016587 · 18801287
3542243	7	239 · 421 · 7547 · 9871 · 33923 · 5464523 · 37287559 · 38128021
3542927	7	127 · 281 · 14813 · 70141 · 647557 · 1287973 · 3381743 · 18911369
3557251	7	29 · 25453 · 47797 · 66067 · 125063 · 273281 · 472697 · 53807629
3589661	7	43 · 1289 · 7393 · 34721 · 140281 · 8216783 · 10542253 · 12375203
3598873	7	113 · 21211 · 48119 · 3727459 · 3738659 · 13788881 · 98035939

p	r	$\Phi_r(p)$
3762137	7	7 · 29 · 113 · 127 · 281 · 743 · 2107771 · 6913229 · 14425811 · 22176337
3863521	7	29 · 1723 · 1301903 · 7967107 · 8714287 · 9929263 · 74163181
3944953	7	29 · 43 · 4336333 · 9937229 · 25858421 · 44103641 · 61506397
3949919	7	7 · 2801 · 36947 · 279679 · 337583 · 983123 · 1253911 · 45042649
4024961	7	239 · 10949 · 25229 · 160343 · 200467 · 462239 · 637729 · 6796763
4480403	7	8821 · 70099 · 208993 · 734021 · 1252483 · 1369607 · 49712419
4505311	7	29 · 337 · 379 · 491 · 2311 · 2437 · 8009 · 547093 · 3532999 · 52742621
4538603	7	29 · 127 · 2129 · 9199 · 220529 · 336211 · 700127 · 775349 · 3010673
4586077	7	43 · 71 · 197 · 21323 · 59333 · 107507 · 251707 · 4919671 · 91842227
4777793	7	379 · 10837 · 134359 · 2972803 · 10304197 · 23262709 · 30248821
4797857	7	7 · 71 · 3221 · 612067 · 1045409 · 12333203 · 20644331 · 46770809
4809781	7	43 · 113 · 12517 · 66851 · 4626511 · 6883451 · 8877499 · 10770761
4860269	7	7 · 659 · 4201 · 6553 · 23857 · 499157 · 726923 · 884171 · 13561507
4900547	7	7 · 1499 · 4159 · 32159 · 181301 · 25164287 · 43296233 · 49962179
4988891	7	1051 · 9857 · 14281 · 1046711 · 28467167 · 45386657 · 77058199
5028323	7	71 · 113 · 281 · 8737 · 12072649 · 15074641 · 58222277 · 77445271
5084423	7	7 · 379 · 1499 · 19979 · 5995081 · 17946349 · 25171567 · 80289007
5198119	7	43 · 281 · 3823 · 33587 · 1579579 · 2488319 · 35659037 · 90721471
5303383	7	7 · 43 · 659 · 743 · 18523 · 653899 · 8876449 · 33188513 · 42308561
5354207	7	29 · 17431 · 1948619 · 2796221 · 3808141 · 28914719 · 77681983
5357977	7	29 · 659 · 2591 · 36527 · 2592199 · 6900853 · 9613129 · 76068427
5418277	7	197023 · 657959 · 6573071 · 28477093 · 29668493 · 35147309
5517643	7	29 · 71 · 1009 · 24977 · 84827 · 162779 · 751997 · 2669507 · 19617977
5538097	7	29 · 43 · 127 · 631 · 18047 · 39103 · 72689 · 150991 · 3284779 · 11348093
5635519	7	7 · 29 · 659 · 63659 · 65479 · 877577 · 1980301 · 2408281 · 13725769
5643667	7	7 · 757 · 11080469 · 11873471 · 15349699 · 30340507 · 99521269
5654071	7	2927 · 7547 · 59333 · 5776457 · 9909593 · 14802173 · 29419237
5696671	7	7 · 181889 · 187139 · 6003859 · 15804461 · 32822371 · 46055129
5710517	7	7 · 337 · 12391 · 226409 · 708989 · 6527641 · 15974477 · 70876499
5848907	7	7 · 43 · 1289 · 1877 · 2969 · 4649 · 185753 · 423109 · 3670969 · 13804673
5899363	7	7 · 43 · 757 · 3221 · 22541 · 66347 · 10123709 · 46354337 · 81837659
5996611	7	29 · 2633 · 7211 · 27077 · 93871 · 17097389 · 23753129 · 81810373
6203371	7	71 · 883 · 22247 · 120947 · 239611 · 1724423 · 15356881 · 53238697
6264107	7	29 · 2521 · 17977 · 37493 · 284509 · 5500153 · 25258367 · 31019927
6414623	7	3067 · 59011 · 190093 · 246289 · 2459801 · 36326963 · 92011039
6450307	7	7253 · 32789 · 33629 · 46327 · 251063 · 331339 · 901279 · 2592829
6459863	7	43 · 127 · 197 · 6203 · 135017 · 963047 · 1548317 · 4360091 · 12405331
6499631	7	4663 · 22247 · 482441 · 751871 · 5595059 · 18452113 · 19406941
6602399	7	239 · 1303 · 2423 · 33013 · 53117 · 86171 · 312509 · 472249 · 4922681
6835589	7	113 · 2801 · 29387 · 30871 · 38459 · 8529949 · 16548211 · 65442931
6957107	7	29 · 71 · 281 · 337 · 7211 · 3943283 · 7611283 · 51182839 · 52498139
6990749	7	379 · 99023 · 356749 · 5165987 · 8074529 · 10425101 · 20047189
7144363	7	449 · 659 · 1429 · 4943 · 6917 · 10739 · 146819 · 169667 · 171823 · 200117
7163917	7	13903 · 232877 · 460013 · 490169 · 2888747 · 3732499 · 17172877
7318039	7	7 · 43 ² · 239 · 4523 · 174931 · 186551 · 280253 · 33381587 · 35957321

p	r	$\Phi_r(p)$
7321123	7	673 · 3389 · 7589 · 8219 · 83609 · 5762597 · 23817487 · 94321109
7434463	7	7 · 29 · 43 · 631 · 1499 · 2633 · 5741 · 20231 · 34147 · 27503813 · 71203441
7532381	7	24179 · 3786119 · 15453887 · 49118371 · 49365779 · 53241889
7580773	7	71 · 127 · 337 · 421 · 11117 · 15541 · 21673 · 39439 · 11804563 · 85103131
8104141	7	19937 · 34217 · 729191 · 1015561 · 1699111 · 5768869 · 57211127
8233787	7	29 · 449 · 1163 · 6091 · 6581 · 1598633 · 6115061 · 7036499 · 7462547
8297413	7	29 · 43 · 113 · 3361 · 35141 · 6164803 · 11216717 · 11364193 · 24951991
8383519	7	29 · 113 · 281 · 1667 · 3221 · 6343 · 40699 · 154351 · 31461487 · 56012111
8399593	7	43 · 197 · 379 · 7589 · 19559 · 55721 · 8675857 · 21248221 · 71744821
8522749	7	463 · 1163 · 46523 · 223273 · 262501 · 2054851 · 11239607 · 11301893
8561057	7	7 · 29 · 1597 · 11131 · 148471 · 429563 · 3604609 · 11776829 · 40297139
8638463	7	7 · 43 · 211 · 421 · 2437 · 39901 · 102593 · 960961 · 16628389 · 97493327
8693947	7	43 · 2591 · 18257 · 20903 · 25747 · 159167 · 355517 · 1188601 · 5864783
8753207	7	7 · 43 · 1499 · 10627 · 12923 · 2138501 · 2663459 · 17963233 · 70944889
8816711	7	7 · 953 · 28183 · 358667 · 4336333 · 6995227 · 13108103 · 17518859
8837557	7	7 · 71 · 127 · 43093 · 785303 · 857669 · 5669441 · 6260297 · 7327139
9194639	7	43 · 1051 · 211933 · 358499 · 501187 · 2049419 · 2232931 · 76726357
9369317	7	43 · 4831 · 42463 · 413197 · 591893 · 1908817 · 2453459 · 66955771
9961531	7	113 · 127 · 3878519 · 47990153 · 59668603 · 62095181 · 98730787
9997441	7	1373 · 20707 · 1089383 · 2475019 · 8841449 · 15675983 · 93978403
10024241	7	127 · 3851 · 9661 · 224071 · 258637 · 2742461 · 28015093 · 48228181
10045507	7	29 · 239 · 491 · 12923 · 127163 · 275087 · 1099729 · 13689061 · 44371111
10225807	7	71 · 2647 · 37199 · 94949 · 763267 · 7842437 · 10000733 · 28773473
10265509	7	71 · 197 · 211 · 3319 · 4271 · 57793 · 73217509 · 77179187 · 85654213
10393237	7	7 · 491 · 6581 · 10151 · 56099 · 133967 · 168281 · 52606471 · 82508287
10567813	7	43 · 127 · 1303 · 7127 · 76819 · 743849 · 2500639 · 4074757 · 47171671
10859309	7	659 · 23269 · 46271 · 71597 · 74761 · 1570339 · 8582617 · 32037461
10869011	7	113 · 1471 · 2003 · 3361 · 67607 · 132833 · 1684229 · 2004787 · 48588401
10906559	7	29 · 883 · 9199 · 759739 · 8829731 · 8910679 · 9742027 · 12270301
11041819	7	43 · 1429 · 12853 · 44647 · 82601 · 90371 · 487481 · 1787633 · 7901251
11171777	7	7 · 29 · 197 · 211 · 673 · 953 · 1373 · 2801 · 8233 · 82279 · 4044503 · 34094551
11413789	7	43 · 393779 · 4251647 · 10500799 · 10717631 · 15261821 · 17880241
11467927	7	449 · 631 · 1429 · 1583 · 2339 · 2927 · 17333 · 497729 · 1843997 · 32586737
11581567	7	211 · 953 · 2801 · 57667 · 1998473 · 27214867 · 30591037 · 44656991
11725339	7	127 · 7757 · 8093 · 153889 · 478171 · 2856547 · 24998849 · 62028779
11741893	7	29 · 379 · 42407 · 188707 · 807731 · 11020003 · 51482383 · 65021783
11930839	7	29 · 43 · 883 · 6581 · 134947 · 2432711 · 3294509 · 8435309 · 43627613
12211543	7	7 · 29 · 43 · 197 · 953 · 4159 · 9542891 · 24530801 · 30381947 · 68407459
12460403	7	43 · 71 · 1373 · 3067 · 6791 · 21911 · 160357 · 1403081 · 1996933 · 4354631
12917647	7	7 · 743 · 2311 · 3347 · 162499 · 831503 · 1611289 · 8021749 · 66130373
13744979	7	29 · 2339 · 189491 · 418181 · 533261 · 1398209 · 17813657 · 94453577
13850567	7	967 · 29723 · 2421959 · 3311519 · 16288021 · 22542479 · 83410783
13977811	7	7 · 43 · 127 · 2731 · 805687 · 6581849 · 11615381 · 18260663 · 63515369
13987403	7	29 · 71 · 127 · 75391 · 109453 · 540233 · 2921339 · 35989423 · 61105087
14125313	7	337 · 659 · 3851 · 17627 · 6981563 · 29563381 · 38003239 · 67172869

p	r	$\Phi_r(p)$
14151341	7	7 · 43 · 281 · 2087 · 6581 · 30241 · 258917 · 3267727 · 5631179 · 47984021
14228573	7	547 · 166363 · 1405153 · 7669411 · 14274107 · 17033311 · 34800893
14257819	7	69623 · 148457 · 244301 · 1530019 · 3633029 · 7253107 · 82518283
14323703	7	29 · 43 · 113 · 127 · 6301 · 325487 · 582821 · 1189651 · 5654293 · 60021529
14409979	7	29 · 71 · 113 · 2857 · 335917 · 568807 · 20740441 · 53427683 · 63614167
14661061	7	29 · 211 · 29401 · 3128791 · 7904653 · 8871199 · 11361113 · 22145593
14830391	7	29 · 43 · 743 · 1163 · 13217 · 88663 · 538511 · 1740187 · 1808843 · 4970659
15225751	7	491 · 2591 · 42337 · 171403 · 275087 · 4882333 · 14201839 · 70752683
15847499	7	43 · 1373 · 2381 · 15401 · 164809 · 343393 · 2048621 · 2231027 · 28286441
16044293	7	43 · 127 · 379 · 449 · 242467 · 9160537 · 12199237 · 12383659 · 54702929
16606193	7	2591 · 1339409 · 1348747 · 2516963 · 2541701 · 10134433 · 69104869
16847081	7	29 · 17851 · 46439 · 255361 · 464927 · 7237567 · 18079181 · 61220083
16955431	7	3319 · 815809 · 1018907 · 3669373 · 10842119 · 11932439 · 18142097
18207781	7	239 · 337 · 1373 · 2689 · 62903 · 314077 · 8550739 · 9063097 · 80032289
18421549	7	29 · 827 · 4271 · 6763 · 295513 · 320083 · 1177751 · 6620587 · 76488343
18443773	7	43 · 71 · 2591 · 29947 · 63799 · 171403 · 5629583 · 36925813 · 73099181
18786679	7	6637 · 852881 · 1694393 · 3354037 · 5747239 · 6603437 · 36010451
18885121	7	113 · 211 · 5153 · 26573 · 189127 · 709913 · 1602553 · 2214983 · 29155169
18892483	7	7 · 449 · 4691 · 3471133 · 16351007 · 18604321 · 37418599 · 78056189
18913991	7	43 · 127 · 28477 · 70393 · 196337 · 604031 · 707071 · 6333419 · 7874819
19646813	7	71 · 10711 · 35491 · 79031 · 3943661 · 7700911 · 14982829 · 59253377
20365361	7	29 · 113 · 13063 · 37423 · 65899 · 308141 · 1545433 · 17212931 · 82444447
20605183	7	29 · 1009 · 241739 · 950251 · 2212631 · 4046239 · 29253001 · 43476413
21101371	7	1471 · 25621 · 29723 · 38767 · 126757 · 15828457 · 20260003 · 50009261
21999787	7	43 · 3767 · 29527 · 96293 · 8901019 · 18195577 · 18254237 · 83265197
22194313	7	7 · 43 · 1877 · 17921 · 116047 · 2935213 · 22445249 · 30195887 · 51134203
22308911	7	29 · 1429 · 2017 · 33811 · 39383 · 49169 · 12581689 · 20851321 · 85862477
22444277	7	659 · 2591 · 19237 · 619921 · 4299653 · 9368059 · 12089351 · 12892027
22489241	7	281 · 6917 · 9787 · 33013 · 33811 · 160091 · 550369 · 1385749 · 49903001
22853483	7	197 · 617 · 6917 · 8933 · 26153 · 275521 · 2499953 · 14378029 · 73238677
22951283	7	29 · 1597 · 2339 · 141961 · 202567 · 234571 · 1711753 · 8649467 · 13510337
23150119	7	50821 · 638359 · 1123403 · 2347549 · 4149881 · 20643421 · 21001177
23893139	7	197 · 3739 · 75377 · 1032697 · 1187551 · 4646671 · 14660521 · 40110883
24066689	7	29 · 71 · 463 · 6217 · 139301 · 4508687 · 12681593 · 55084261 · 74726429
25032031	7	43 · 337 · 1303 · 1583 · 4831 · 9157 · 1525763 · 1818293 · 3183559 · 21066809
27287233	7	7 · 43 · 113 · 617 · 631 · 13469 · 17921 · 21701 · 1564417 · 38480653 · 98861057
27568511	7	239 · 3557 · 7477 · 11299 · 31193 · 464213 · 2266237 · 3274811 · 56880671
28208827	7	71 · 491 · 5783 · 15401 · 489133 · 523573 · 3009203 · 4658557 · 45202361
28234649	7	1163 · 8807 · 161561 · 500459 · 1078757 · 2027411 · 6028457 · 46399151
28361629	7	7477 · 292531 · 640949 · 3785531 · 29562317 · 54942413 · 60379747
28379903	7	1051 · 2143 · 11117 · 645877 · 685063 · 15504413 · 46640651 · 65215781
28578083	7	1362551 · 27985189 · 46422433 · 53190257 · 66210901 · 87383227
29211373	7	43 · 743 · 1583 · 3613 · 367739 · 669173 · 8397901 · 17979781 · 91510819
29609431	7	29 · 127 · 463 · 497281 · 704131 · 1087717 · 1562107 · 24689519 · 26903143
29695933	7	7 · 29 · 43 · 5573 · 15443 · 116201 · 809173 · 5336563 · 21595351 · 84240997

p	r	$\Phi_r(p)$
30314399	7	15443 · 199697 · 2437219 · 2907577 · 5152463 · 69163459 · 99649061
30693889	7	43 · 379 · 463 · 2017 · 6091 · 168617 · 1045549 · 1093331 · 1261387 · 37100743
30823421	7	2003 · 6553 · 25579 · 249859 · 2857709 · 3456377 · 14080837 · 73505153
31029787	7	43 · 127 · 337 · 1289 · 15443 · 55931 · 117937 · 7935047 · 19252423 · 24179569
31307207	7	7 · 29 · 197 · 60397 · 84673 · 164599 · 1302449 · 1853713 · 2389591 · 4848299
31839151	7	7 · 43 · 281 · 379 · 5573 · 8807 · 28463 · 120401 · 428639 · 15778337 · 28567757
32244859	7	29 · 281 · 3137 · 26293 · 136949 · 3212413 · 10808771 · 13802909 · 25478083
33729187	7	65003 · 240899 · 662369 · 3182341 · 21742757 · 27410237 · 74850161
33804481	7	71 · 127 · 211 · 281 · 11159 · 212297 · 496427 · 4835153 · 21772969 · 22544593
35236219	7	43 · 2647 · 370133 · 379849 · 5516827 · 16272901 · 23886157 · 55775567
35893283	7	239 · 162499 · 269851 · 377231 · 583031 · 609407 · 34740581 · 43819049
37620101	7	7 · 29 · 113 · 169373 · 198073 · 202567 · 271181 · 2365399 · 2422421 · 11702909
37843849	7	7 · 127 · 379289 · 10479673 · 11367077 · 16158563 · 61939991 · 73068367
38475977	7	7 · 29 · 197 · 547 · 953 · 2843 · 93871 · 2171737 · 5175493 · 7029611 · 7380689
38859413	7	71 · 197 · 463 · 96167 · 137369 · 2263381 · 3597749 · 66699137 · 74105137
39434663	7	3557 · 3613 · 8387 · 14281 · 19181 · 292223 · 832721 · 22835639 · 22922047
39890003	7	43 · 673 · 2017 · 4019 · 27329 · 114157 · 223273 · 697397 · 1695611 · 20850103
40106387	7	29 · 8233 · 236293 · 948053 · 2849351 · 9070559 · 39311917 · 76583893
40178219	7	29 · 659 · 6329 · 388991 · 516839 · 1106029 · 4427417 · 5061281 · 6979967
40188523	7	43 · 127 · 631 · 1429 · 1493759 · 4569811 · 28565671 · 56512723 · 77644211
41111381	7	29^2 · 2843 · 3823 · 6217 · 500333 · 838153 · 1867951 · 1881811 · 57635551
41442329	7	29 · 43 · 379 · 1289 · 61979 · 748567 · 894419 · 950867 · 6640663 · 31736489
41981081	7	3221 · 19237 · 140057 · 503231 · 2538803 · 3414181 · 8413021 · 17189131
42075751	7	127 · 2003 · 35393 · 54139 · 243517 · 390391 · 3172471 · 5671499 · 6655097
42523937	7	7 · 6917 · 7127 · 63617 · 169709 · 587063 · 1477043 · 30176749 · 60652663
42560369	7	113 · 127 · 281 · 1597 · 3739 · 12503 · 14813 · 43051 · 281233 · 3492077 · 31520483
43305763	7	211 · 379 · 2591 · 58031 · 111833 · 185123 · 8455147 · 32955749 · 95091641
44316197	7	239 · 281 · 42239 · 72101 · 375509 · 940003 · 2141749 · 6141059 · 7977271
44894383	7	1933 · 14869 · 26209 · 150893 · 702913 · 27240151 · 41178523 · 91355993
46242337	7	7 · 71 · 631 · 659 · 3347 · 34763 · 121577 · 298733 · 313909 · 698111 · 51089501
46268653	7	29 · 701 · 1303 · 16871 · 169177 · 2642333 · 15521101 · 46904327 · 67461437
47971057	7	7 · 197 · 4019 · 16339 · 1534219 · 2120917 · 30030043 · 33295109 · 41363813
48298753	7	29 · 43 · 421 · 897373 · 2083957 · 2444569 · 2787443 · 23082067 · 82209541
49854929	7	43 · 883 · 24977 · 77267 · 516433 · 1036267 · 1126343 · 8647913 · 40199209
49937521	7	113 · 4691 · 10711 · 31333 · 48091 · 574393 · 3163469 · 19285169 · 51728041
50058647	7	29^2 · 5783 · 78989 · 2830073 · 4965563 · 7082377 · 17306381 · 23779757
50837341	7	1163 · 2843 · 23899 · 30059 · 56393 · 1055881 · 3963317 · 5527481 · 5571343
51418093	7	29 · 631 · 3137 · 6917 · 9157 · 166027 · 384259 · 787879 · 4589999 · 22029673
51706331	7	43 · 211 · 379 · 279511 · 1264649 · 2441293 · 7497043 · 20739853 · 41417867
54797191	7	7 · 71 · 757 · 1093 · 2689 · 102551 · 4918411 · 20991811 · 31632763 · 73102933
55067269	7	113 · 1723 · 2549 · 195161 · 579629 · 882631 · 2717821 · 4843693 · 42747223
55739507	7	421 · 1429 · 11173 · 59921 · 78653 · 411083 · 3030371 · 26621561 · 28545749
55959779	7	7 · 29 · 2689 · 5867 · 53089 · 60383 · 219437 · 10421951 · 14671147 · 89147843
56260723	7	7 · 14057 · 68881 · 87641 · 96601 · 853091 · 1161007 · 6951883 · 80263877
56414999	7	29 · 71 · 54881 · 158621 · 276277 · 372107 · 4841593 · 39829987 · 90723011

p	r	$\Phi_r(p)$
57038833	7	43 · 113 · 15877 · 54979 · 103307 · 4871861 · 5050417 · 50920381 · 62728961
58284521	7	7 · 125399 · 394409 · 445019 · 2522087 · 21093199 · 63569549 · 75240257
58776539	7	43 · 1471 · 11411 · 150893 · 155821 · 920473 · 5183221 · 6154751 · 82737593
59329663	7	7 · 113 · 2843 · 4271 · 36037 · 136403 · 2823857 · 3085741 · 7503763 · 14128451
59708479	7	127 · 967 · 7001 · 25117 · 381739 · 546967 · 4843693 · 22607411 · 91770463
59903947	7	337 · 3361 · 21491 · 74131 · 99191 · 443689 · 2097257 · 16206233 · 17119579
61118989	7	7 · 4243 · 4831 · 69497 · 462911 · 1901131 · 3386741 · 19301059 · 90867827
61762193	7	29 · 281 · 463 · 757 · 2087 · 3011 · 259813 · 1083881 · 1155127 · 2018899 · 4709069
61964431	7	2521 · 9829 · 168869 · 787879 · 1712383 · 2880739 · 57176701 · 60875039
64632527	7	7 · 29 · 71 · 113 · 449 · 107857 · 237707 · 1054649 · 2712179 · 14250167 · 95387419
64687097	7	29 · 43 · 7127 · 85751 · 1098511 · 1147301 · 10887367 · 81337649 · 86138501
66606157	7	281 · 5153 · 6301 · 26293 · 38977 · 41609 · 50821 · 502699 · 666821 · 13173889
68113807	7	10613 · 30829 · 83791 · 574813 · 3701881 · 10213001 · 12842411 · 13051697
68441137	7	71 · 28001 · 62539 · 3685501 · 4812851 · 18304511 · 26277259 · 96890977
71014399	7	7 · 197 · 281 · 1093 · 10753 · 56533 · 6121543 · 17289007 · 59463251 · 79154657
72331219	7	43 · 28547 · 644197 · 2722931 · 8324471 · 8465297 · 22430381 · 42076049
73553561	7	29 · 24473 · 29023 · 33013 · 39971 · 3750839 · 4043341 · 4434571 · 86625211
74306809	7	3851 · 38921 · 604031 · 662719 · 1496167 · 3987229 · 15379967 · 30578689
75682331	7	29 · 113 · 2143 · 136963 · 2045359 · 4599533 · 12094321 · 34224121 · 50172767
75790223	7	113 · 3347 · 17599 · 27847 · 288583 · 509363 · 11523359 · 13190311 · 45766351
76085351	7	29 · 127 · 17333 · 92107 · 1504651 · 1612759 · 5960809 · 36097979 · 63189491
77118799	7	113 · 275339 · 532267 · 1731731 · 2435203 · 9050329 · 9605737 · 34647761
77580787	7	29 · 281 · 491 · 659 · 2297 · 6917 · 9437 · 804161 · 1985677 · 3533293 · 97748449
77720411	7	71 · 113 · 7687 · 108739 · 1192171 · 4204523 · 5063129 · 15639079 · 82803001
77827181	7	197 · 337 · 1877 · 138923 · 400681 · 2568119 · 7625227 · 39191363 · 41744207
78106307	7	281 · 1289 · 36779 · 332473 · 949621 · 1351099 · 1483763 · 2085287 · 12913181
85891943	7	71 · 337 · 911 · 839791 · 1201201 · 2164121 · 2627143 · 43202587 · 74344019
87885949	7	491 · 631 · 12853 · 225569 · 791519 · 1592753 · 2320207 · 2597113 · 67529659
88761593	7	43 · 71 · 239 · 911 · 6763 · 61153 · 70589 · 129403 · 1190897 · 6944813 · 23546923
89669737	7	239 · 757 · 1855981 · 4543813 · 10344223 · 12682027 · 26326301 · 98653213
90107203	7	197 · 421 · 39971 · 79997 · 145601 · 4639979 · 5009327 · 9937789 · 60012331
90196549	7	29 · 239 · 1093 · 8429 · 39047 · 60103 · 105211 · 7074523 · 53210123 · 90721667
92892127	7	29 · 1093 · 1933 · 18439 · 748609 · 13756877 · 25182221 · 35846021 · 61175311
93058573	7	10459 · 16927 · 399281 · 512597 · 1225183 · 9568931 · 35313391 · 43292201
95303477	7	29 · 281 · 673 · 967 · 16493 · 159097 · 1333193 · 11940041 · 41532961 · 81442901
95709899	7	127 · 3011 · 5503 · 11047 · 11971 · 60271 · 130579 · 1964243 · 3893191 · 45895459
96397919	7	7001 · 35533 · 314693 · 627901 · 5457971 · 5471887 · 9429869 · 57964453
97023863	7	71 · 7309 · 21169 · 33461 · 325487 · 1758947 · 3357901 · 13230953 · 89221679
97073359	7	967 · 2017 · 29723 · 109201 · 213599 · 288317 · 2990051 · 20861779 · 34407199
97185839	7	29 · 71 · 11593 · 189743 · 570851 · 13680311 · 19892419 · 29525861 · 40559359
97330193	7	29 · 113 · 1093 · 810643 · 895231 · 4341191 · 26141809 · 34000723 · 84760103
97330283	7	7 · 43 · 113 · 7477 · 9437 · 24977 · 42379 · 1958419 · 2292949 · 6886727 · 10821259
99129991	7	29 · 44549 · 220151 · 3705101 · 15917371 · 19970147 · 29313551 · 96639061

A.6 Programs

A.6.1 cubeprg.gp

```
\\
\\ cubeprg.gp
\\
\\ by Takeshi GOTO (Oct. 2005)
\\ URL: http://www.ma.noda.tus.ac.jp/u/tg/perfect.html
\\
\\ used to find all pairs (P,Q,R) satisfying
\\  $Q^3 \mid \Phi_R(P)$ ,  $P, Q < 10^8$ ,  $6680 \leq R < 5 \times 10^7$ .
\\ output files: mysqrs.txt, mycubes.txt, numchkd.txt
\\
prmdiv(a)=
{
  local(p);
  p=2;
  while(p<=a,
    if(a%p==0,return(p));
    p=nextprime(p+1)
  )
}

modpow(a,b,p)=
{
  local(x);
  x=Mod(a,p)^b;
  return(lift(x))
}

fngen(q)=
{
  local(a,qq,p,c);
  a=2; c=0;
  while(c==0,
    qq=q-1;c=1;
    while(qq>1,
      p=prmdiv(qq);
      if(modpow(a,(q-1)/p,q)==1,c=0);
      while(qq%p==0,qq=qq/p)
    );
    if(c==0,a=a+1)
  );
  if(modpow(a,q-1,q^2)==1,a=a+q);
  return(a)
}
```

```

check(f,t)=
{
  local(a,g,q,r);
  r=nextprime(f);
  if(r>t,return("Not a valid interval.));
  while(r<=t,
    print("Checking for R= ",r);
    q=2*r+1;
    while(q<10^8,
      if(isprime(q),
        g=Mod(fngen(q),q^2);
        a=g^(q*(q-1)/r); p=1;
        for(i=1,r-1,
          p=lift(p*a);
          if(p<10^2, write("mysqrs.txt",q,"^2 divides ",p,"^",r,"-1"),
            if(p<10^8 && isprime(p) && modpow(p,r,q^3)==1,
              print(q,"^3 divides ",p,"^",r,"-1");
              write("mycubes.txt",q,"^3 divides ",p,"^",r,"-1")
            )
          )
        )
      );
      q=q+2*r
    );
    r=nextprime(r+1)
  );
  write("numchkd.txt","Checked R= ",f," to R= ",t)
}

```

A.6.2 cubeprg.ub

```

10  '
20  ' CUBEPRG.UB
30  '
40  ' originally produced by Paul M. Jenkins
50  ' modified by Takeshi GOTO (Oct. 2005)
60  ' URL: http://www.ma.noda.tus.ac.jp/u/tg/perfect.html
70  '
80  ' This file enable us to find all pairs (P,Q,R) satisfying
90  '  $Q^3 \mid \Phi_R(P)$ ,  $P, Q < 10^8$ ,  $6680 \leq R < 5 \times 10^7$ .
100 ' We did it, using GP script "cubeprg.gp".
110 ' output files: MYSQRS.TXT, MYCUBES.TXT, NUMCHKD.TXT
120 ' expected CPU time: about two hours for an interval of length  $10^4$ 
130 '
140 input "Begin checking at";F

```

```

150 input "Up to";T
160 clr time
170 R=nxtprm(F-1)
180 if R>T then print "Not a valid interval":end
190 print "Checking for R=";R
200 Q=2*R+1
210 while Q<100000000
220     if prmdiv(Q)<Q then goto 380
230     G=fnGEN(Q)
240     A=modpow(G,Q*(Q-1)//R,Q^2):P=1
250     for I=1 to R-1
260         P=(P*A)@(Q^2) 'Note that if R>5000, then Q^2>10^8
270         if P<100 then
280             :print=print+"mysqrs.txt"
290             :print Q;"^2 divides";P;"^";R;"-1"
300             :print=print
310         endif
320         if and{P<100000000,prmdiv(P)=P,modpow(P,R,Q^3)=1} then
330             :print=print+"mycubes.txt"
340             :print Q;"^3 divides";P;"^";R;"-1"
350             :print=print
360         endif
370     next
380     Q=Q+2*R
390 wend
400 R=nxtprm(R):if R<=T then goto 190
410 print=print+"numchkd.txt"
420 print "Checked R=";F;"to R=";T;"in";time
430 print=print
440 end
450 '
460 ' a generator of  $(\mathbb{Z}/Q^2\mathbb{Z})^*$ 
470 '
480 fnGEN(Q)
490 local A,QQ,P
500 A=2
510 QQ=Q-1
520 while QQ>1
530     P=prmdiv(QQ)
540     if modpow(A,(Q-1)//P,Q)=1 then A=A+1:goto 510
550     while QQ@P=0:QQ=QQ//P:wend
560 wend
570 if modpow(A,Q-1,Q^2)=1 then A=A+Q
580 return(A)

```

A.6.3 sqrprg.ub

```
10  '
20  ' SQRPRG.UB
30  '
40  ' originally produced by Paul M. Jenkins
50  ' modified by Takeshi GOTO (Oct. 2005)
60  ' URL: http://www.ma.noda.tus.ac.jp/u/tg/perfect.html
70  '
80  ' used to check for cubic or square divisors of  $\Phi_R(P)$ 
90  ' for  $2503 < R \leq 6679$ 
100 ' output files: MYCUBES.TXT, NUMCHKD.TXT
110 ' expected CPU time: about an hour for  $2503 < R \leq 6679$ 
120 '
130 input "Begin checking at";F
140 input "Up to";T:clr time
150 open "memo" as file1(10000) word 100:C=1
160 R=nxtprm(F-1)
170 if R>T then print "Not a valid interval":end
180 print "Checking for R=";R
190 Q=2*R+1:C=1
200 while Q<10^8
210   if prmdiv(Q)<Q then goto 460
220   G=fnGEN(Q)
230   A=modpow(G,Q*(Q-1)//R,Q^2):P=1
240   for I=1 to R-1
250     P=(P*A)@(Q^2)
260     while P<10^8
270       if or{prmdiv(P)<P,P<10^2} then goto 430
280       for I=1 to C-1
290         if P=file1(I) then
300           :print=print+"mysqrs.txt"
310           :print "More than one square divides";P;"^";R;"-1"
320           :print=print
330           :cancel for
340           :goto 370
350         endif
360       next
370       file1(C)=P:C=C+1
380       if modpow(P,R,Q^3)=1 then
390         :print=print+"mycubes.txt"
400         :print Q;"^3 divides";P;"^";R;"-1"
410         :print=print
420       endif
430       P=P+Q^2
440     wend
450   next
```

```

460     Q=Q+2*R
470     wend
480     R=nxtprm(R):if R<=T then goto 180
490     print=print+"numchkd.txt"
500     print "Checked R=";F;"to R=";T;"in";time
510     print=print
520     close:kill "memo+"
530     end
540     '
550     ' a generator of  $(\mathbb{Z}/Q^2\mathbb{Z})^*$ 
560     '
570     fnGEN(Q)
580     local A,QQ,P
590     A=2
600     QQ=Q-1
610     while QQ>1
620         P=prmdiv(QQ)
630         if modpow(A,(Q-1)//P,Q)=1 then A=A+1:goto 600
640         while QQ@P=0:QQ=QQ//P:wend
650     wend
660     if modpow(A,Q-1,Q^2)=1 then A=A+Q
670     return(A)

```

A.6.4 rvalue.ub

```

10     '
20     ' RVALUE.UB
30     '
40     ' originally produced by Paul M. Jenkins as "Prop8.ub"
50     ' modified by Takeshi GOTO (Oct. 2005)
60     ' URL: http://www.ma.noda.tus.ac.jp/u/tg/perfect.html
70     '
80     ' used to find the values R(P) for P<100
90     ' output file: RVALUE.TXT
100    ' expected CPU time: about thirty minutes
110    '
120    Epsilon=10^5*#Eps ' an upper bound of round-off error
130    P=3
140    print=print+"rvalue.txt"
150    while P<100
160        R=4500
170        while R<5*10^4
180            Q=1
190            C=8*log(10)+log(R)
200            while Q<10^8
210                Q=Q+2*R

```

```

220     if prmdiv(Q)=Q then C=C+log(Q)
230     wend
240     if C+Epsilon>(R-1)*log(P) then
250         :print R
260         :if C<=(R-1)*log(P) then print "check again"
270     endif
280     R=nxtprm(R)
290     wend
300     print "for P=";P
310     P=nxtprm(P)
320     wend
330     print=print

```

A.6.5 claim1.ub

```

10     '
20     ' CLAIM1.UB
30     '
40     ' originally produced by Paul M. Jenkins as "Prop5.ub"
50     ' modified by Takeshi GOTO (Oct. 2005)
60     ' URL: http://www.ma.noda.tus.ac.jp/u/tg/perfect.html
70     '
80     ' used to show Claim1
90     ' expected CPU time: about one minute
100    '
110    Epsilon=105*#eps ' an upper bound of round-off error
120    R=6007
130    while R<5*104
140        Q=1
150        C=8*log(10)+0.5*log(R)
160        while Q<108
170            Q=Q+2*R
180            if prmdiv(Q)=Q then C=C+log(Q)
190        wend
200        if C+Epsilon>(R-1)*log(10) then print R
210        R=nxtprm(R)
220    wend

```

A.6.6 claim2.ub

```

10     '
20     ' CLAIM2.UB
30     '
40     ' originally produced by Paul M. Jenkins as "Prop6.ub"
50     ' modified by Takeshi GOTO (Oct. 2005)
60     ' URL: http://www.ma.noda.tus.ac.jp/u/tg/perfect.html

```

```

70  '
80  ' used to show Claim2
90  ' expected CPU time: less than one minute
100 '
110 Epsilon=105*#eps ' an upper bound of round-off error
120 R=nxtprm(4000)
130 while R<6680
140   Q=1
150   C=8*log(10)+log(R)
160   while Q<108
170     Q=Q+2*R
180     if prmdiv(Q)=Q then C=C+log(Q)
190   wend
200   if C+Epsilon>2*(R-1)*log(10) then print R
210   R=nxtprm(R)
220 wend

```

A.6.7 claim3.ub

```

10  '
20  ' CLAIM3.UB
30  '
40  ' originally produced by Paul M. Jenkins as "Prop6.ub"
50  ' modified by Takeshi GOTO (Oct. 2005)
60  '   URL: http://www.ma.noda.tus.ac.jp/u/tg/perfect.html
70  '
80  ' used to show Claim3
90  ' expected CPU time: less than one minute
100 '
110 Epsilon=105*#eps ' an upper bound of round-off error
120 R=nxtprm(2500)
130 while R<4724
140   Q=1
150   C=16*log(10)+log(R)
160   while Q<108
170     Q=Q+2*R
180     if prmdiv(Q)=Q then C=C+log(Q)
190   wend
200   if C+Epsilon>6*(R-1)*log(10) then print R
210   R=nxtprm(R)
220 wend

```

A.6.8 claim4.ub

```

10  '
20  ' CLAIM4.UB

```

```

30  '
40  ' originally produced by Paul M. Jenkins as "Prop6.ub"
50  ' modified by Takeshi GOTO (Oct. 2005)
60  '   URL: http://www.ma.noda.tus.ac.jp/u/tg/perfect.html
70  '
80  ' used to show Claim4
90  ' expected CPU time: less than one minute
100 '
110 Epsilon=105*#eps ' an upper bound of round-off error
120 R=nxtprm(2000)
130 while R<2708
140   Q=1
150   C=16*log(10)+log(R)
160   while Q<108
170     Q=Q+2*R
180     if prmdiv(Q)=Q then C=C+log(Q)
190   wend
200   if C+Epsilon>7*(R-1)*log(10) then print R
210   R=nxtprm(R)
220 wend

```

A.6.9 accept.ub

```

10  '
20  ' ACCEPT.UB
30  '
40  ' by Takeshi GOTO (Oct. 2005)
50  '   URL: http://www.ma.noda.tus.ac.jp/u/tg/perfect.html
60  '
70  ' used to find acceptable values  $\Phi_R(P)$  for  $7 \leq R \leq 4723$ 
80  ' output file: ACCEPT.TXT
90  ' expected CPU time: about three hours for one value of R
100 '
110 input "Begin checking at";F
120 input "Up to";T
130 R=nxtprm(F-1)
140 if R>T then print "Not a valid interval":end
150 print
160 '
170 clr time:clr C:clr C1
180 print=print+"accept.txt"
190 print "Checking for R=";R
200 print=print
210 '
220 ' write all pairs (P,Q) satisfying  $Q \mid \Phi_R(P)$  and  $P, Q < 10^8$ 
230 ' on file1

```



```

240 open "pairs.ubd" for create as file1(2^32-1) word 8
250 Q=2*R+1
260 while Q<100000000
270   if prmdiv(Q)<Q then goto 400
280   print "                               ";chr(13);
290   print " Q=";Q;
300   G=fnGEN(Q)
310   W=modpow(G,(Q-1)//R,Q):P=1
320   for I%=1 to R-1
330     P=(P*W)@Q
340     while P<100000000
350       if or{prmdiv(P)<P,P=2} then goto 370
360       C=C+1:file1(C)=pack(P,Q)
370       P=P+Q
380     wend
390   next
400   Q=Q+2*R
410 wend
420 '
430 ' sort file1
440 file1(C+1)=pack(0,0) ' dummy data
450 print:print "Now sorting"
460 print=print+"accept.txt"
470 print "( # of elements=";C;)"
480 print=print
490 open "sort.ubd" for create as file2(100000) word 8
500 gosub *Sort(1,C,0)
510 print:print "# of not sorted intervals=";C1
520 C2=C1
530 for I=1 to C2
540   gosub *Sort(member(file2(I),1),member(file2(I),2),0)
550 next
560 if C1>C2 then
570   :print=print+"accept.txt"
580   :print "Not sorted yet"
590   :print=print
600 endif
610 '
620 ' list possibly acceptable values
630 print
640 print "Now checking"
650 print=print+"accept.txt"
660 D=1
670 while D<=C
680   P=member(file1(D),1)
690   V=0

```

```

700     if P@R=1 then V=log(R)
710     while member(file1(D),1)=P
720         Q=member(file1(D),2)
730         V=V+log(Q)
740         N%=2
750         while modpow(P,R,Q^N%)=1
760             V=V+log(Q)
770             N%=N%+1
780         wend
790         D=D+1
800     wend
810     if V>(R-1)*log(P) then print P
820 wend
830 '
840 '   closing
850 print "Checked R=";R;"in";time
860 print
870 print=print
880 close:kill "pairs.ubd":kill "sort.ubd"
890 R=nxtprm(R)
900 if R<=T then goto 170
910 end
920 '
930 '   quick sort
940 '
950 '   Remark. N represents the number of recursion levels.
960 '   There is a limit of N on UBASIC,
970 '   so we need some steps to sort data.
980 '
990 *Sort(A,B,N)
1000 local I,J,M,TMP
1010 if A>=B then goto 1320
1020 if N>48 then
1030     :C1=C1+1
1040     :file2(C1)=pack(A,B)
1050     :goto 1320
1060 endif
1070 print "                               ";chr(13);
1080 print A;
1090 I=A
1100 '
1110 '   determine a key element
1120 M=member(file1(I),1)
1130 while and{I<=B,member(file1(I),1)=M}:I=I+1:wend
1140 if I>B then goto 1320
1150 if member(file1(I),1)>M then M=member(file1(I),1)

```

```

1160  '
1170  I=A:J=B
1180  while I<J
1190      while member(file1(I),1)<M:I=I+1:wend
1200      while member(file1(J),1)>=M:J=J-1:wend
1210      '
1220      ' switch the elements
1230      if I<J then
1240          :TMP=file1(I)
1250          :file1(I)=file1(J)
1260          :file1(J)=TMP
1270      endif
1280      '
1290  wend
1300  gosub *Sort(A,I-1,N+1)
1310  gosub *Sort(J+1,B,N+1)
1320  return
1330  '
1340  ' generator of (Z/QZ)^*
1350  '
1360  fnGEN(Q)
1370  local A,QQ,P
1380  A=2
1390  QQ=Q-1
1400  while QQ>1
1410      P=prmdiv(QQ)
1420      if modpow(A,(Q-1)//P,Q)=1 then A=A+1:goto 1390
1430      while QQ@P=0:QQ=QQ//P:wend
1440  wend
1450  return(A)

```

A.6.10 accept2.ub

```

10  '
20  ' ACCEPT2.UB
30  '
40  ' originally produced by Paul M. Jenkins as "PROP9.UB"
50  ' modified by Takeshi GOTO (Oct. 2005)
60  ' URL: http://www.ma.noda.tus.ac.jp/u/tg/perfect.html
70  '
80  ' used to find acceptable values  $\Phi_R(P)$  for  $P < 100$ 
90  ' output file: ACCEPT2.TXT
100 ' expected CPU time: about ten minutes for one value of P
110 '
120 V=0
130 input "Enter P";Start

```

```

140 P=nxtprm(Start-1)
150 input "Enter Rp";Rp
160 clr time
170 print=print+"accept2.txt"
180 R=7
190 while R<Rp
200     V=0
210     if P@R=1 then V=V+log(R)
220     X=2*R+1
230     while X<100000000
240         if prmdiv(X)<X then goto 310
250         if modpow(P,R,X)=1 then V=V+log(X)
260             :if modpow(P,R,X^2)=1 then V=V+log(X)
270                 :if modpow(P,R,X^3)=1 then V=V+log(X)
280                     :print X;"^3 divides";P;"^";R;"-1"
290                         :endif
300                             :endif
310                                 endif
320                                     X=X+2*R
330                                         wend
340                                             if V>=(R-1)*log(P) then
350                                                 :print "Look at P=";P;"and R=";R
360                                                     endif
370                                                         R=nxtprm(R)
380                                                             wend
390 print "Checked P=";P;"for Rp <";Rp;"in";time
400 print=print
410 end

```

A.6.11 ptester.ub

```

10  '
20  ' PTESTER.UB
30  '
40  ' originally produced by Paul M. Jenkins
50  ' modified by Takeshi GOTO (Oct. 2005)
60  '   URL: http://www.ma.noda.tus.ac.jp/u/tg/perfect.html
70  '
80  ' used to calculate P^*
90  ' expected CPU time: about thirty minutes
100 '
110 point -8
120 Pcount=0
130 Pproduct=1
140 P=41
150 while P<10^8

```

```

160     print P;
170     print "                ";chr(13);
180     Pcount=Pcount+1
190     Pproduct=Pproduct*P/(P-1)
200     P=nxtprm(P)
210 wend
220 print "Pcount=";Pcount
230 print "Pproduct=";Pproduct
240 end

```

A.6.12 stester.ub

```

10  '
20  ' STESTER.UB
30  '
40  ' originally produced by Paul M. Jenkins
50  ' modified by Takeshi GOTO (Oct. 2005)
60  '   URL: http://www.ma.noda.tus.ac.jp/u/tg/perfect.html
70  '
80  ' used to calculate S^*
90  ' expected CPU time: about ten minutes
100 '
110 point -8
120 Scount=0
130 Sproduct=1
140 N=41
150 while N<10^8
160     if and{N@3>1 and N@5>1} then
170         :print N;
180         :print "                ";chr(13);
190         :Scount=Scount+1
200         :Sproduct=Sproduct*N/(N-1)
210     endif
220     N=nxtprm(N)
230 wend
240 print "Scount=";Scount
250 print "Sproduct=";Sproduct
260 end

```

A.6.13 ttester.ub

```

10  '
20  ' TTESTER.UB
30  '
40  ' originally produced by Paul M. Jenkins
50  ' modified by Takeshi GOTO (Oct. 2005)

```

```

60 ' URL: http://www.ma.noda.tus.ac.jp/u/tg/perfect.html
70 '
80 ' used to calculate T^*
90 ' expected CPU time: about five minutes
100 '
110 point -8
120 Tcount=0
130 Tproduct=1
140 N=41
150 while N<10^8
160   if and{N@3=1,N@5=1} then
170     :print N;
180     :print "                               ";chr(13);
190     :Tcount=Tcount+1
200     :Tproduct=Tproduct*N/(N-1)
210   endif
220   N=nxtprm(N)
230 wend
240 print "Tcount=";Tcount
250 print "Tproduct=";Tproduct
260 end

```

A.6.14 utester.ub

```

10 '
20 ' UTESTER.UB
30 '
40 ' by Takeshi GOTO (Oct. 2005)
50 ' URL: http://www.ma.noda.tus.ac.jp/u/tg/perfect.html
60 '
70 ' used to calculate U^*
80 ' expected CPU time: about two hours
90 '
100 point -8
110 clr time
120 Ucount=0
130 Uproduct=1
140 '
150 ' write all pairs (P,Q) satisfying
160 ' P \equiv 1 (mod 3), P \not\equiv 1 (mod 5),
170 ' Q | \Phi_5(P) and P,Q<10^8 on file1
180 open "pairs.ubd" for create as file1(2^32-1) word 8
190 Q=11
200 while Q<100000000
210   if prmdiv(Q)<Q then goto 340
220   print "                               ";chr(13);

```

```

230   print " Q=";Q;
240   G=fnGEN(Q)
250   W=modpow(G,(Q-1)//5,Q):P=1
260   for I%=1 to 4
270     P=(P*W)@Q
280     while P<100000000
290       if or{prmdiv(P)<P,P<40,P@3<>1,P@5=1} then goto 310
300       C=C+1:file1(C)=pack(P,Q)
310       P=P+Q
320     wend
330   next
340   Q=Q+10
350 wend
360 '
370 ' sort file1
380 file1(C+1)=pack(0,0) ' dummy data
390 print:print "Now sorting ( # of elements=";C;"          "
400 open "sort.ubd" for create as file2(100000) word 8
410 gosub *Sort(1,C,0)
420 print:print "# of not sorted intervals=";C1
430 C2=C1
440 for I=1 to C2
450   gosub *Sort(member(file2(I),1),member(file2(I),2),0)
460 next
470 if C1>C2 then print "Not sorted yet"
480 '
490 print
500 D=1
510 P=43
520 while P<100000000
530   if or{prmdiv(P)<P,P@5=1} then goto 700
540   V=1
550   if member(file1(D),1)>P then goto 660
560   while member(file1(D),1)=P
570     Q=member(file1(D),2)
580     V=V*Q
590     N%=2
600     while modpow(P,5,Q^N%)=1
610       V=V*Q
620       N%=N%+1
630     wend
640     D=D+1
650   wend
660   if V<P^4+P^3+P^2+P+1 then
670     :Ucount=Ucount+1
680     :Uproduct=Uproduct*P/(P-1)

```

```

690     endif
700     P=P+6
710 wend
720 '
730 '   closing
740 print "Ucount=";Ucount
750 print "Uproduct=";Uproduct
760 print "Calculated in";time
770 close:kill "pairs.ubd":kill "sort.ubd"
780 end
790 '
800 '   quick sort
810 '
820 '   Remark. N represents the number of recursion levels.
830 '   There is a limit of N on UBASIC,
840 '   so we need some steps to sort data.
850 '
860 *Sort(A,B,N)
870 local I,J,M,TMP
880 if A>=B then goto 1190
890 if N>48 then
900   :C1=C1+1
910   :file2(C1)=pack(A,B)
920   :goto 1190
930 endif
940 print "                               ";chr(13);
950 print A;
960 I=A
970 '
980 '   determine a key element
990 M=member(file1(I),1)
1000 while and{I<=B,member(file1(I),1)=M}:I=I+1:wend
1010 if I>B then goto 1190
1020 if member(file1(I),1)>M then M=member(file1(I),1)
1030 '
1040 I=A:J=B
1050 while I<J
1060   while member(file1(I),1)<M:I=I+1:wend
1070   while member(file1(J),1)>=M:J=J-1:wend
1080   '
1090   '   switch the elements
1100   if I<J then
1110     :TMP=file1(I)
1120     :file1(I)=file1(J)
1130     :file1(J)=TMP
1140   endif

```



```

1150     '
1160 wend
1170 gosub *Sort(A,I-1,N+1)
1180 gosub *Sort(J+1,B,N+1)
1190 return
1200     '
1210     ' generator of  $(Z/QZ)^*$ 
1220     '
1230 fnGEN(Q)
1240 local A,QQ,P
1250 A=2
1260 QQ=Q-1
1270 while QQ>1
1280     P=prmdiv(QQ)
1290     if modpow(A,(Q-1)//P,Q)=1 then A=A+1:goto 1260
1300     while QQ@P=0:QQ=QQ//P:wend
1310 wend
1320 return(A)

```

A.6.15 vtester.ub

```

10     '
20     ' VTESTER.UB
30     '
40     ' by Takeshi GOTO (Oct. 2005)
50     ' URL: http://www.ma.noda.tus.ac.jp/u/tg/perfect.html
60     '
70     ' used to calculate  $V^*$ 
80     ' expected CPU time: about an hour
90     '
100 point -8
110 clr time
120 Vcount=0
130 Vproduct=1
140     '
150     ' write all pairs (P,Q) satisfying
160     '  $P \not\equiv 1 \pmod{3}$ ,  $P \equiv 1 \pmod{5}$ ,
170     '  $Q \mid \Phi_3(P)$  and  $P,Q < 10^8$  on file1
180 open "pairs.ubd" for create as file1(2^32-1) word 8
190 Q=7
200 while Q<100000000
210     if prmdiv(Q)<Q then goto 340
220     print "                ";chr(13);
230     print " Q=";Q;
240     G=fnGEN(Q)
250     W=modpow(G,(Q-1)//3,Q):P=1

```

```

260     for I%=1 to 2
270         P=(P*W)@Q
280         while P<100000000
290             if or{prmdiv(P)<P,P<40,P@3=1,P@5<>1} then goto 310
300             C=C+1:file1(C)=pack(P,Q)
310             P=P+Q
320         wend
330     next
340     Q=Q+6
350 wend
360 '
370 ' sort file1
380 file1(C+1)=pack(0,0) ' dummy data
390 print:print "Now sorting ( # of elements=";C;" "
400 open "sort.ubd" for create as file2(100000) word 8
410 gosub *Sort(1,C,0)
420 print:print "# of not sorted intervals=";C1
430 C2=C1
440 for I=1 to C2
450     gosub *Sort(member(file2(I),1),member(file2(I),2),0)
460 next
470 if C1>C2 then print "Not sorted yet"
480 '
490 print
500 D=1
510 P=41
520 while P<100000000
530     if or{prmdiv(P)<P,P@3=1} then goto 700
540     V=1
550     if member(file1(D),1)>P then goto 660
560     while member(file1(D),1)=P
570         Q=member(file1(D),2)
580         V=V*Q
590         N%=2
600         while modpow(P,3,Q^N%)=1
610             V=V*Q
620             N%=N%+1
630         wend
640         D=D+1
650     wend
660     if V<P^2+P+1 then
670         :Vcount=Vcount+1
680         :Vproduct=Vproduct*P/(P-1)
690     endif
700     P=P+10
710 wend

```

```

720  '
730  '  closing
740  print "Vcount=";Vcount
750  print "Vproduct=";Vproduct
760  print "Calculated in";time
770  close:kill "pairs.ubd":kill "sort.ubd"
780  end
790  '
800  '  quick sort
810  '
820  '  Remark. N represents the number of recursion levels.
830  '  There is a limit of N on UBASIC,
840  '  so we need some steps to sort data.
850  '
860  *Sort(A,B,N)
870  local I,J,M,TMP
880  if A>=B then goto 1190
890  if N>48 then
900    :C1=C1+1
910    :file2(C1)=pack(A,B)
920    :goto 1190
930  endif
940  print "                ";chr(13);
950  print A;
960  I=A
970  '
980  '  determine a key element
990  M=member(file1(I),1)
1000 while and{I<=B,member(file1(I),1)=M}:I=I+1:wend
1010 if I>B then goto 1190
1020 if member(file1(I),1)>M then M=member(file1(I),1)
1030 '
1040 I=A:J=B
1050 while I<J
1060   while member(file1(I),1)<M:I=I+1:wend
1070   while member(file1(J),1)>=M:J=J-1:wend
1080   '
1090   '  switch the elements
1100   if I<J then
1110     :TMP=file1(I)
1120     :file1(I)=file1(J)
1130     :file1(J)=TMP
1140   endif
1150   '
1160 wend
1170 gosub *Sort(A,I-1,N+1)

```

```

1180 gosub *Sort(J+1,B,N+1)
1190 return
1200 '
1210 ' generator of (Z/QZ)^*
1220 '
1230 fnGEN(Q)
1240 local A,QQ,P
1250 A=2
1260 QQ=Q-1
1270 while QQ>1
1280   P=prmdiv(QQ)
1290   if modpow(A,(Q-1)//P,Q)=1 then A=A+1:goto 1260
1300   while QQ@P=0:QQ=QQ//P:wend
1310 wend
1320 return(A)

```

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