Introduction to Arthur's multiplicity formula

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Doi–Naganuma lifting

- $F = \mathbb{Q}(\sqrt{D})$: real quadratic field (with narrow class number 1);
- χ_D: the Dirichlet character associated to $F/\mathbb{Q}$;
- $S_{k,k}(\mathrm{SL}_2(\mathcal{O}_F)) = \{$Hilbert modular cusp forms of level one$\}$.

Theorem (Doi–Naganuma, 1969)

For any Hecke eigenform $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$,
there exists a Hecke eigenform $\hat{f} \in S_{k,k}(\mathrm{SL}_2(\mathcal{O}_F))$ such that

$$L(s, \hat{f}) = L(s, f) L(s, f, \chi_D).$$

The proof used Weil's converse theorem for Hilbert modular forms.

Automorphic forms

How do we understand the Doi–Naganuma lifting? $\rightarrow$ Use rep. theory.

For $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$, define an aut. form $\varphi_f \colon \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \rightarrow \mathbb{C}$ by

$$\varphi_f(\gamma g \kappa) = (f|_k g)(\sqrt{-1}),$$

for $\gamma \in \mathrm{GL}_2(\mathbb{Q})$, $g \in \mathrm{GL}_2(\mathbb{R})^+$, $\kappa \in \mathrm{GL}_2(\widehat{\mathbb{Z}})$.
It generates an irr. cusp. unitary aut. rep. τ_f of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$.

If F has narrow class number one, for $\hat{f} \in S_{k,k}(\mathrm{SL}_2(\mathcal{O}_F))$,
we can define $\varphi_{\hat{f}} \colon \mathrm{GL}_2(\mathbb{A}_F) \rightarrow \mathbb{C}$ and $\tau_{\hat{f}}$ of $\mathrm{GL}_2(\mathbb{A}_F)$ similarly.

Godement–Jacquet L-function

All representations are over $\mathbb{C}$.

- F: number field.

Let τ be an irr. adm. rep. of $\mathrm{GL}_n(\mathbb{A}_F)$. $\rightsquigarrow \tau \cong \otimes'_v \tau_v$ s.t.
τ_v irr. adm. rep. of $\mathrm{GL}(F_v)$, τ_v is unramified for almost all $v < \infty$.
$\rightsquigarrow$ Godement–Jacquet L-function $L(s,\tau) = \prod_v L(s,\tau_v)$.

If τ_v unramified $\rightsquigarrow \{\alpha_{v,1}, \ldots, \alpha_{v,n}\} \subset \mathbb{C}$: Satake parameter
$\rightsquigarrow L(s,\tau_v) = \prod_{i=1}^n (1 - \alpha_{v,i} q_v^{-s})^{-1}$.

Bound for Satake parameter

If τ is cuspidal unitary and τ_v is unramified, then $q_v^{-\frac{1}{2}} < |\alpha_{v,i}| < q_v^{\frac{1}{2}}$.

Langlands conjecture

The *Langlands Problem* would classify aut. rep's. of $\mathrm{GL}_n(\mathbb{A}_F)$.

Langlands Problem (1967, 1970)

We wish the existence of a locally compact group $\mathcal{L}_F$,
which is "close to" $\mathrm{Gal}(\overline{F}/F)$ or the Weil group W_F, such that

$$\{\text{irr. cusp. aut. rep's. } \tau \text{ of } \mathrm{GL}_n(\mathbb{A}_F)\} \overset{1:1}{\longleftrightarrow} \{n\text{-dim. irr. rep's. } \phi \text{ of } \mathcal{L}_F\}$$

satisfying $L(s,\tau) = L(s,\phi)$, and some additional properties.

The group $\mathcal{L}_F$ is called the *hypothetical Langlands group*.

Base Change

As Galois groups, there should be $\mathcal{L}_{\mathbb{Q}(\sqrt{D})} \hookrightarrow \mathcal{L}_{\mathbb{Q}}$. Hence

$$
\begin{array}{ccccc}
f & \rightsquigarrow & [\tau \text{ of } \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})] & \longleftrightarrow & [\phi\colon \mathcal{L}_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})] \\
\text{DN lift} \downarrow & & \downarrow & & \downarrow \mathrm{Res} \\
\hat{f} & \leftsquigarrow & [\hat{\tau} \text{ of } \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}(\sqrt{D})})] & \longleftrightarrow & [\hat{\phi}\colon \mathcal{L}_{\mathbb{Q}(\sqrt{D})} \to \mathrm{GL}_2(\mathbb{C})]
\end{array}
$$

Generalization: For an extension E/F, the *Base change lifting*

$$\mathrm{BC}\colon \{\text{irr. aut. rep's. of } \mathrm{GL}_n(\mathbb{A}_F)\} \to \{\text{irr. aut. rep's. of } \mathrm{GL}_n(\mathbb{A}_E)\}$$

is expected, and proven for E/F cyclic (Langlands, Arthur–Clozel).

Compatibility with the Local Langlands Correspondence

The local Langlands group $\mathcal{L}_{F_v}$ is the *Weil–Delige group*

$$\mathcal{L}_{F_v} := \begin{cases} W_{F_v} & \text{if } F_v \text{ is archimedean,} \\ W_{F_v} \times \mathrm{SU}(2) & \text{if } F_v \text{ is non-archimedean.} \end{cases}$$

The LLC for GL_n is established by Langlands, Harris–Taylor, Henniart.

Local-Global compatibility

There should be an embedding $\mathcal{L}_{F_v} \hookrightarrow \mathcal{L}_F$ such that

$$\begin{array}{ccc} [\tau = \otimes_v \tau_v \text{ of } \mathrm{GL}_n(\mathbb{A}_F)] & \longleftrightarrow & [\phi\colon \mathcal{L}_F \to \mathrm{GL}_n(\mathbb{C})] \\ \downarrow & & \downarrow \mathrm{Res} \\ [\tau_v \text{ of } \mathrm{GL}_n(F_v)] & \xleftrightarrow{\mathrm{LLC}} & [\phi_v\colon \mathcal{L}_{F_v} \to \mathrm{GL}_n(\mathbb{C})]. \end{array}$$

Generalized Ramanujan conjecture

When $f = \sum_{n>0} a_f(n)q^n \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ is a normalized Hecke eigenform, Deligne (1974) proved $|a_f(p)| \leq 2p^{\frac{k-1}{2}}$ (Ramanujan conjecture).
$\rightsquigarrow$ The Satake parameter $\{\alpha_p, \alpha_p^{-1}\}$ of $\tau_{f,p}$ satisfies $|\alpha_p| = 1$ since

$$1 - a_f(p)X + p^{k-1}X^2 = (1 - \alpha_p p^{\frac{k-1}{2}} X)(1 - \alpha_p^{-1} p^{\frac{k-1}{2}} X).$$

$\rightsquigarrow$ The irr. rep. $\tau_{f,p}$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ is tempered.

Generalized Ramanujan conjecture

It $\tau = \otimes_v \tau_v$ is an irr. cusp. unitary aut. rep. of $\mathrm{GL}_n(\mathbb{A}_F)$, then τ_v is tempered for any v.

Generalized Ramanujan v.s. Langlands conjecture

It is hoped that τ of $\mathrm{GL}_n(\mathbb{A}_F)$ is unitary $\iff$ $\phi(\mathcal{L}_F)$ is bounded.
By LLC, τ_v of $\mathrm{GL}_n(F_v)$ is tempered $\iff$ $\phi_v(\mathcal{L}_{F_v})$ is bounded.

$$
\begin{array}{ccc}
\tau \text{ of } \mathrm{GL}_n(\mathbb{A}_F) \text{ is unitary} & \overset{\text{Langlands}}{\underset{??}{\Leftarrow = \Rightarrow}} & \phi(\mathcal{L}_F) \text{ is bounded} \\
\text{Generalized Ramanujan} \Big\Downarrow & & ?? \Big\Downarrow \text{Local-Grobal} \\
\tau_v \text{ of } \mathrm{GL}_n(F_v) \text{ is tempered} & \Longleftrightarrow & \phi_v(\mathcal{L}_{F_v}) \text{ is bounded}
\end{array}
$$

Observation

The existence of the hypothetical Langlands group $\mathcal{L}_F$ implies the generalized Ramanujan conjecture.

What a big problem the existence of the hypothetical Langlands group is!

Discrete spectrum

Theorem (Mœglin–Waldspurger, 1989)

Let π be in $L^2_{\mathrm{disc}}(\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_F)^1) \subset L^2(\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_F)^1)$.
Then π is the unique irr. quotient of the parabolically induced rep.

$$|\cdot|^{\frac{d-1}{2}}\tau \times |\cdot|^{\frac{d-3}{2}}\tau \times \cdots \times |\cdot|^{-\frac{d-1}{2}}\tau$$

for some $d \mid n$ and an irr. cusp. unitary aut. rep. τ of $\mathrm{GL}_{n/d}(\mathbb{A}_F)$.

It is hoped that τ of $\mathrm{GL}_{n/d}(\mathbb{A}_F)$ is unitary iff $\phi(\mathcal{L}_F)$ is bounded. Hence

$$\{\text{irr. aut. rep's. } \pi \text{ in } L^2_{\mathrm{disc}}(\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_F)^1)\}$$

$$\updownarrow \; 1{:}1$$

$$\{n\text{-dim. irr. rep's. } \psi = \phi \boxtimes S_d \text{ of } \mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C}) \text{ s.t. } \psi(\mathcal{L}_F) \text{ bdd.}\}.$$

Symplectic group

A symplectic group is given by

$$\mathrm{Sp}_{2n}(\mathbb{R}) = \left\{ g \in \mathrm{GL}_{2n}(\mathbb{R}) \;\middle|\; {}^t g \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix} \right\}.$$

Define the Siegel upper half space by

$$\mathfrak{H}_n = \left\{ Z \in \mathrm{Mat}_n(\mathbb{C}) \;\middle|\; {}^t Z = Z,\ \mathrm{Im}(Z) > 0 \right\}.$$

The Lie group $\mathrm{Sp}_{2n}(\mathbb{R})$ acts on $\mathfrak{H}_n$ by

$$g \langle Z \rangle = (AZ + B)(CZ + D)^{-1}, \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix},\ Z \in \mathfrak{H}_n.$$

Siegel modular form

A hol. function $F\colon \mathfrak{H}_n \to \mathbb{C}$ is a *holomorphic Siegel cusp form of weight k* if $(F|_k\gamma)(Z) = F(Z)$ for $\gamma \in \mathrm{Sp}_{2n}(\mathbb{Z})$ and $Z \in \mathfrak{H}_n$, where

$$(F|_k g)(Z) \coloneqq \det(CZ+D)^{-k} F(\gamma\langle Z\rangle), \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{Z}),$$

and a cusp condition.
Write

$$S_k(\mathrm{Sp}_{2n}(\mathbb{Z})) = \{\text{holomorphic Siegel cusp form of weight } k\}.$$

As well as the case of $n = 1$, there is a Hecke theory.

From modular forms to automorphic forms

A Hecke eigenform $F \in S_k(\mathrm{Sp}_{2n}(\mathbb{Z}))$ gives
a cusp form $\varphi_F \colon \mathrm{Sp}_{2n}(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$ by

$$\varphi_F(\gamma g \kappa) = (F|_k g)(\sqrt{-1} \cdot \mathbf{1}_n)$$

for $\gamma \in \mathrm{Sp}_{2n}(\mathbb{Q})$, $g \in \mathrm{Sp}_{2n}(\mathbb{R})$, $\kappa \in \mathrm{Sp}_{2n}(\widehat{\mathbb{Z}})$.
It generates an irr. cusp. aut. rep. $\pi_F = \otimes'_v \pi_{F,v}$ of $\mathrm{Sp}_{2n}(\mathbb{A}_\mathbb{Q})$.

Goal

To classify $F \in S_k(\mathrm{Sp}_{2n}(\mathbb{Z}))$, or π_F of $\mathrm{Sp}_{2n}(\mathbb{A}_\mathbb{Q})$.

Local classification

Before global, recall the local classification.

- $\mathcal{L}_{\mathbb{Q}_v}$: local Langlands group of $\mathbb{Q}_v$ (Weil or Weil–Deligne group).

Local Langlands correspondence (Langlands 1989, Arthur 2013)

There is a canonical surjection

$$\mathrm{Irr}(\mathrm{Sp}_{2n}(\mathbb{Q}_v)) \to \{\phi_v \colon \mathcal{L}_{\mathbb{Q}_v} \to \mathrm{SO}_{2n+1}(\mathbb{C})\} / \cong$$

such that

1. the fiber Π_{ϕ_v} of ϕ_v is finite, and called an *L-packet*;
2. $\pi_v \in \Pi_{\phi_v}$ is tempered $\iff$ $\phi_v(\mathcal{L}_{\mathbb{Q}_v})$ is bounded.

Global classification?

As GL_n-case, one might naively hope that
there exists a surjection with commutative diagram

$$\begin{array}{ccc} \{\text{irr. cusp. aut. rep's. of } \mathrm{Sp}_{2n}(\mathbb{A}_{\mathbb{Q}})\} & \overset{??}{\dashrightarrow} & \{\phi\colon \mathcal{L}_{\mathbb{Q}} \to \mathrm{SO}_{2n+1}(\mathbb{C})\} \\ \Big\downarrow{\scriptstyle \text{Localization}} & & \Big\downarrow{\scriptstyle \text{Res}} \\ \mathrm{Irr}(\mathrm{Sp}_{2n}(\mathbb{Q}_v)) & \longrightarrow & \{\phi_v\colon \mathcal{L}_{\mathbb{Q}_v} \to \mathrm{SO}_{2n+1}(\mathbb{C})\}. \end{array}$$

But, if it were true,
any irr. cusp. unitary aut. rep's. $\pi = \otimes'_v \pi_v$ of $\mathrm{Sp}_{2n}(\mathbb{A}_F)$ would satisfy the Ramanujan conjecture, i.e., π_v would be tempered.

Saito–Kurokawa lifting

However, the Ramanujan conjecture does not hold in general. For example:

Saito–Kurokawa lifting (1977–1981)

For a Hecke eigenform $f \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ with odd k,
there exists a Hecke eigenform $F_{\mathrm{SK}} \in S_{k+1}(\mathrm{Sp}_4(\mathbb{Z}))$ such that

$$L(s, F_{\mathrm{SK}}, \mathrm{std}) = \zeta(s)L(s+k-1, f)L(s+k, f).$$

For the irr. cusp. aut. rep. $\pi_{\mathrm{SK}} = \otimes'_v \pi_{\mathrm{SK},v}$ generated by $\varphi_{F_{\mathrm{SK}}}$,
its local factor $\pi_{\mathrm{SK},p}$ is non-tempered for any $p < \infty$.

How should we understand? (Why is k odd?)

Arthur's $\mathrm{SL}_2(\mathbb{C})$

Describe not the cuspidal spectrum but the discrete spectrum.
Hinted by the result on the discrete spectrum for $\mathrm{GL}_N(\mathbb{A}_F)$:

Arthur conjecture (1989)

$$\{\text{irr. aut. rep's. } \pi \text{ in } L^2_{\mathrm{disc}}(\mathrm{Sp}_{2n}(\mathbb{Q})\backslash\mathrm{Sp}_{2n}(\mathbb{A}_\mathbb{Q}))\}$$

$$\Big\downarrow ??$$

$$\{\psi\colon \mathcal{L}_\mathbb{Q} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{2n+1}(\mathbb{C}) \text{ s.t. "elliptic" and } \psi(\mathcal{L}_\mathbb{Q}) \text{ bdd.}\}.$$

If $f \in S_{2k}(\mathrm{SL}_2(\mathbb{Z})) \rightsquigarrow \tau_f$ of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \rightsquigarrow \phi_f\colon \mathcal{L}_\mathbb{Q} \to \mathrm{SL}_2(\mathbb{C})$,
then π_{SK} should correspond to

$$(\phi_f \boxtimes S_2) \oplus \mathbf{1}\colon \mathcal{L}_\mathbb{Q} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_5(\mathbb{C}).$$

Local A-packets

As the local analogue, associated to a *local A-parameter*

$$\psi_v \colon \mathcal{L}_{\mathbb{Q}_v} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{2n+1}(\mathbb{C}) \quad \text{s.t. } \psi_v(\mathcal{L}_{Q_v}) \text{ bdd.},$$

Arthur himself construct a multi-set Π_{ψ_v} over $\mathrm{Irr}_{\mathrm{unit}}(\mathrm{Sp}_{2n}(\mathbb{Q}_v))$, which is called the *local A-packet* (c.f., the talks of Oshima and Oi).

Theorem (Mœglin (2006), Arancibia–Mœglin–Renard (2018), Mœglin–Renard (to appear), Arancibia–Mezo (in progress))

The local A-packet Π_{ψ_v} is multiplicity-free,
i.e., it is in fact a subset of $\mathrm{Irr}_{\mathrm{unit}}(\mathrm{Sp}_{2n}(\mathbb{Q}_v))$.

Arthur's multiplicity formula

Associated to a "*global A-parameter*" $\psi\colon \mathcal{L}_{\mathbb{Q}} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{2n+1}(\mathbb{C})$, we define the *global A-packet* by the restricted tensor product

$$\Pi_\psi = \otimes'_v \Pi_{\psi_v}.$$

"Theorem" (Arthur's multiplicity formula, 2013)

For each ψ, there exists a subset $\Pi_\psi(\varepsilon_\psi) \subset \Pi_\psi$ such that

$$L^2_{\mathrm{disc}}(\mathrm{Sp}_{2n}(\mathbb{Q})\backslash \mathrm{Sp}_{2n}(\mathbb{A}_{\mathbb{Q}})) \cong \bigoplus_{\psi} \bigoplus_{\pi \in \Pi_\psi(\varepsilon_\psi)} \pi.$$

Interpretation of the definition of ψ

Recall that $\psi\colon \mathcal{L}_{\mathbb{Q}} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{2n+1}(\mathbb{C})$.

Problem

How should we define ψ without using $\mathcal{L}_{\mathbb{Q}}$?

Idea

Use an expectation

$$\{n\text{-dim. irr. rep's. } \phi \text{ of } \mathcal{L}_{\mathbb{Q}}\} \xleftrightarrow{1:1} \{\text{irr. cusp. aut. rep's. } \tau \text{ of } \mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})\}$$

as a building block of global A-parameters.

c.f., Morimoto's talk.

Holomorphic cusp forms

Define $\mathcal{S}_k(\mathrm{Sp}_{2n}(\mathbb{A}_\mathbb{Q}))$ as the image of

$$S_k(\mathrm{Sp}_{2n}(\mathbb{Z})) \ni F \mapsto \varphi_F \in L^2_{\mathrm{disc}}(\mathrm{Sp}_{2n}(\mathbb{Q})\backslash \mathrm{Sp}_{2n}(\mathbb{A}_\mathbb{Q})).$$

Then

$$\mathcal{S}_k(\mathrm{Sp}_{2n}(\mathbb{A}_\mathbb{Q})) \cong \bigoplus_{\psi} \bigoplus_{\pi \in \Pi_\psi(\varepsilon_\psi)} \left(\pi \cap \mathcal{S}_k(\mathrm{Sp}_{2n}(\mathbb{A}_\mathbb{Q}))\right).$$

Problem

When is there $\pi \in \Pi_\psi(\varepsilon_\psi)$ such that $\pi \cap \mathcal{S}_k(\mathrm{Sp}_{2n}(\mathbb{A}_\mathbb{Q})) \neq 0$?

We consider 3 Steps.

Archimedean condition

By the holomorphy condition and weight condition of $S_k(\mathrm{Sp}_n(\mathbb{Z}))$, if $\pi = \otimes'_v \pi_v \in \Pi_\psi$ satisfies $\pi \cap \mathcal{S}_k(\mathrm{Sp}_{2n}(\mathbb{A}_\mathbb{Q})) \neq 0$, then π_∞ is the *lowest weight module* $\mathcal{D}_k^{(n)}$ of $\mathrm{Sp}_{2n}(\mathbb{R})$ of weight k, which imposes strong conditions on ψ_∞.

- If $k > n$, then $\mathcal{D}_k^{(n)}$ is discrete series.
  In this case, ψ_∞ is so-called *Adams–Johnson*, which is easier (c.f., Oshima's talk).
- If $k = n$, then $\mathcal{D}_k^{(n)}$ is a limit of discrete series.
  If $k < n$, then $\mathcal{D}_k^{(n)}$ is not tempered.
  These cases are very difficult.

Non-archimedean condition

By the level one condition of $S_k(\mathrm{Sp}_n(\mathbb{Z}))$,
if $\pi = \otimes'_v \pi_v \in \Pi_\psi$ satisfies $\pi \cap \mathcal{S}_k(\mathrm{Sp}_{2n}(\mathbb{A}_\mathbb{Q})) \neq 0$,
then π_p is an *unramified* rep. of $\mathrm{Sp}_{2n}(\mathbb{Q}_p)$ for any p,
which implies $\psi_p \colon \mathcal{L}_{\mathbb{Q}_p} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{2n+1}(\mathbb{C})$ is *unramified*.

Theorem (Mœglin, 2009)

If ψ_p is unramified, then Π_{ψ_p} contains a unique unramified rep. π_p^0.
Moreover, if $\psi_p = \oplus_i(\phi_i \boxtimes S_{d_i})$ with $\dim(S_d) = d$, then

$$L(s, \pi_p^0, \mathrm{std}) = \prod_i \prod_{j=1}^{d_i} L\left(s + \frac{d_i + 1}{2} - j, \phi_i\right).$$

c.f., Oi's talk.

Sign condition

If $\mathcal{D}_k^{(n)} \in \Pi_{\psi_\infty}$ and if ψ_p is unramified with $\pi_p^0 \in \Pi_{\psi_p}$ for all p,
then $\pi = \mathcal{D}_k^{(n)} \otimes (\otimes'_p \pi_p^0) \in \Pi_\psi$.
Next problem is *whether* $\pi \in \Pi_\psi(\varepsilon_\psi)$ *or not (i.e.,* π *is automorphic or not)*.

Example: Ikeda lifting

Let $f \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ be associated with $\phi_f \colon \mathcal{L}_\mathbb{Q} \to \mathrm{SL}_2(\mathbb{C})$. Consider

$$\psi_f^{(2n)} = (\phi_f \boxtimes S_{2n}) \oplus \mathbf{1} \colon \mathcal{L}_\mathbb{Q} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{4n+1}(\mathbb{C}).$$

Then $\pi_p^0 \in \Pi_{\psi_{f,p}^{(2n)}}$ and $\mathcal{D}_{k+n}^{(2n)} \in \Pi_{\psi_{f,\infty}^{(2n)}}$ if $k > n$. Moreover,

$$\pi = \mathcal{D}_{k+n}^{(2n)} \otimes (\otimes'_p \pi_p^0) \in \Pi_\psi(\varepsilon_\psi) \iff k \equiv n \bmod 2.$$

A list of applications

There are several applications of Arthur's multiplicity formula:

- Classification of modular forms (in particular, existence of liftings).
- The (strong) multiplicity one theorem for $S_k(\mathrm{Sp}_{2n}(\mathbb{Z}))$.
- The dimension of $S_k(\mathrm{Sp}_{2n}(\mathbb{Z}))$.
- The rigidity property.
- Towards the Harder conjecture.

Thank you very much.