Twisted doubling integrals for classical groups

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Automorphic L-functions, I

- F: number field with adele ring $\mathbb{A}$
- G: reductive group over F
- π: irreducible cuspidal automorphic representation of $G(\mathbb{A})$
- $\pi = \otimes'_v \pi_v$: restricted tensor product, where π_v is unramified for almost all places
- ${}^L G$: L-group of G
- $\rho : {}^L G \to \mathrm{GL}_n(\mathbb{C})$.

Automorphic L-functions, II

For an unramified place:

- Satake isomorphism: π_v unramified $\leftrightarrow$ Satake parameter t_v (a semi-simple conjugacy class in $G^\vee$).
- $q_v = \#\mathcal{O}_v/\mathcal{P}_v$
- Local L-function:

$$L_v(s, \pi_v, \rho) = \frac{1}{\det(I - \rho(t_v)q_v^{-s})}.$$

Fix a finite set of places such that π_v is unramified if $v \notin S$. Define global partial L-function:

$$L^S(s, \pi, \rho) = \prod_{v \notin S} L_v(s, \pi_v, \rho).$$

This is compatible with the Langlands correspondence in Atobe's talk.

Automorphic L-functions, III

Basic question

Show that $L^S(s, \pi, \rho)$ admits meromorphic continuation to $\mathbb{C}$, and has a functional equation for $s \mapsto 1 - s$.

Basic method

Find a global integral that represents the desired L-function.

To obtain an Euler product, almost all examples use some kind of multiplicity one results.

Examples

- Godement-Jacquet integrals: matrix coefficients (so ... works for all cuspidal representations)
- Rankin-Selberg integrals for $\mathrm{GL}_m \times \mathrm{GL}_n$ (Jacquet – Piatetski-Shapiro – Rallis): uniqueness of Whittaker models
- Langlands-Shahidi method: uniqueness of Whittaker models
- Doubling integrals (Piatetski-Shapiro – Rallis): matrix coefficients and ...

Automorphic L-functions, IV

This talk

A generalization of the doubling integrals – twisted doubling integrals.
Main references:

- arXiv 1710.00905 (with Friedberg, Ginzburg and Kaplan)
- arXiv 1908.10298

We will discuss the obstructions that arise when extending these constructions.

The doubling zeta integrals, I

- $\mathcal{W} = (W, \langle\ ,\ \rangle)$: a quadratic space over F
- W: vector space over F of dimension n
- $\langle\ ,\ \rangle$: non-degenerate bilinear form on W
- $G = G(\mathcal{W})$: isometry group of $\mathcal{W}$.

Examples: $\mathrm{Sp}_{2n}, \mathrm{O}_n$.

The doubling zeta integrals, II

The doubling map

Define $\mathcal{W}^{\square} = (W^{\square}, \langle\ ,\ \rangle^{\square})$ where

$$W^{\square} = W_{+} \oplus W_{-}$$

and

$$\langle (x_{+}, x_{-}), (y_{+}, y_{-}) \rangle^{\square} = \langle x_{+}, y_{+} \rangle - \langle x_{-}, y_{-} \rangle.$$

Define $G^{\square} = G(\mathcal{W}^{\square})$.
The group $G \times G$ acts on $W^{\square}$ via

$$(g_1, g_2) \cdot (x_{+}, x_{-}) = (g_1 x_{+}, g_2 x_{-}).$$

This gives a homomorphism

$$\iota : G \times G \to G^{\square}.$$

The doubling zeta integrals, III

Siegel parabolic subgroup

Define
$$W^{\Delta} = \{(x, x) \in W^{\square} : x \in W\}.$$

Then $\langle\ ,\ \rangle^{\square}|_{W^{\Delta}\times W^{\Delta}} = 0$, i.e. W^{Δ} is a totally isotropic subspace. This gives a Siegel parabolic subgroup $P(W^{\Delta}) = M(W^{\Delta})N(W^{\Delta})$ or $P = MN$.

Eisenstein series

- $\chi : F^{\times}\backslash\mathbb{A}^{\times} \to \mathbb{C}^{\times}$
- $\chi \circ \det$ is a character of $\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A})$; this gives a character of $M(F)\backslash M(\mathbb{A})$
- $I(s, \chi) = \mathrm{Ind}_{P(\mathbb{A})}^{G^{\square}(\mathbb{A})}(\chi \circ \det) \cdot \delta_P^s$
- $f^{(s)} \in I(s, \chi)$, one attaches an Eisenstein series

$$E(f^{(s)})(g) = \sum_{\gamma\in P(F)\backslash G^{\square}(F)} f^{(s)}(\gamma g).$$

The doubling integrals, IV

- π: irreducible cuspidal representation of $G(\mathbb{A})$
- $\xi_1 \in \pi$ and $\xi_2 \in \pi^\vee$

Global zeta integral

$Z(\xi_1 \boxtimes \xi_2, f^{(s)}) =$

$$\int_{G(F)\backslash G(\mathbb{A})\times G(F)\backslash G(\mathbb{A})} \xi_1(g_1)\xi_2(g_2)E(f^{(s)})(\iota(g_1,g_2))\, dg_1\, dg_2.$$

The doubling integrals, V

Unfolding

$$Z(\xi_1 \boxtimes \xi_2, f^{(s)}) = \int\limits_{G(\mathbb{A})} \mathcal{P}(\pi(g)\xi_1 \boxtimes \xi_2) f^{(s)}(\iota(g, e))\ dg,$$

where

$$\mathcal{P}(\xi_1 \boxtimes \xi_2) = \int\limits_{G(F)\backslash G(\mathbb{A})} \xi_1(g)\xi_2(g)\ dg.$$

This is an Euler product since $\mathcal{P}$ is decomposable and local components of one-dimensional representations are one-dimensional.

Unramified calculation

$$Z(\xi_1 \boxtimes \xi_2, f^{(s)}) \approx L(s, \pi \times \chi).$$

The twisted doubling integrals, I

Goal: tensor product L-function $G \times \mathrm{GL}_k$.

The doubling map

- $\mathcal{W}^{\square,k} = (W^{\square,k}, \langle\ ,\ \rangle^{\square,k})$ where
- $W^{\square,k} = W_1^{\square} \oplus \cdots \oplus W_k^{\square}$
- $\langle\ ,\ \rangle^{\square,k} = \langle\ ,\ \rangle_1^{\square} \oplus \cdots \oplus \langle\ ,\ \rangle_k^{\square}$.
- $G^{\square,k} = G(\mathcal{W}^{\square,k})$
- $(g_1, g_2) \in G \times G$ acts on $W^{\square,k}$ via

$$
\begin{aligned}
(g_1, g_2)\cdot(x_{1+}, x_{1-}, x_{2+}, x_{2-}, \cdots, x_{k+}, x_{k-}) \\
= (g_1 x_{1+}, g_2 x_{1-}, g_1 x_{2+}, g_1 x_{2-}, \cdots, g_1 x_{k+}, g_1 x_{k-}).
\end{aligned}
$$

This defines a homomorphism $\iota_k : G \times G \to G^{\square,k}$.

The twisted doubling integrals, II

Siegel parabolic subgroup

Define

$$W^{\Delta,k} = W_1^{\Delta} \oplus \cdots \oplus W_k^{\Delta}.$$

Define $P = P(W^{\Delta,k}) \subset G^{\square,k}$

Fourier coefficient in the orbit $((2k-1)^n 1^n)$

In one moment. This does not appear when $k = 1$.

Eisenstein series

Given τ on the group $\mathrm{GL}_k(\mathbb{A})$, want to define a representation on $\mathrm{GL}_{kn}(\mathbb{A})$ which looks like

$$\tau \circ \det .$$

This appears in Atobe's talk – the generalized Speh representations.

Degenerate Whittaker models/coefficients, I

Nilpotent orbits

Let $\mathcal{N}_{\mathfrak{g}}$ be the set of nilpotent elements in a semisimple Lie algebra $\mathfrak{g}$. Under the adjoint action, it becomes a disjoint union of nilpotent orbits.

- There is a partial order on the set of nilpotent orbits
- GL_n case: the theory of Jordan canonical form
- Nilpotent orbits of GL_n are in bijection with partitions of n.
- For classical groups, nilpotent orbits are in bijection with partitions with additional assumptions.

Degenerate Whittaker models/coefficients, II

Examples

The orbit (3^2):

$$\begin{pmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & & 0 & & \\ & & & 1 & 0 & \\ & & & & 1 & 0 \end{pmatrix} \text{ or } f_{(3^2)} = \begin{pmatrix} 0 & 0 & & & & \\ 0 & 0 & & & & \\ 1 & 0 & 0 & 0 & & \\ 0 & 1 & 0 & 0 & & \\ & & 1 & 0 & 0 & 0 \\ & & 0 & 1 & 0 & 0 \end{pmatrix}$$

Observe: image of

$$\mathrm{GL}_2 \to \mathrm{GL}_6, \qquad g \mapsto \mathrm{diag}(g, g, g)$$

lies in the stabilizer of $f_{(3^2)}$.

Generalization: for the orbit (k^n), its stabilizer contains the image of

$$\mathrm{GL}_n \to \mathrm{GL}_{kn}, \qquad g \mapsto \mathrm{diag}(g, g, \cdots, g).$$

Degenerate Whittaker models/coefficients, III

Example: orbit $(3^2 1^2)$

$$\begin{pmatrix} 0 &&&&&&& \\ 1 & 0 &&&&&& \\ & 1 & 0 &&&&& \\ &&& 0 &&&& \\ &&& 1 & 0 &&& \\ &&&& 1 & 0 && \\ &&&&&& 0 & \\ &&&&&&& 0 \end{pmatrix} \text{ or } f_{(3^2 1^2)} = \begin{pmatrix} 0 & 0 &&&&&& \\ 0 & 0 &&&&&& \\ 1 & 0 & 0 & 0 &&&& \\ 0 & 1 & 0 & 0 &&&& \\ && 1 & 0 & 0 & 0 && \\ && 0 & 1 & 0 & 0 && \\ &&&&&& 0 & 0 \\ &&&&&& 0 & 0 \end{pmatrix}$$

Note: the image

$$\mathrm{GL}_2 \times \mathrm{GL}_2 \to \mathrm{GL}_8, \qquad (g_1, g_2) \mapsto \mathrm{diag}(g_1, g_1, g_1, g_2)$$

lies in the stabilizer of $f_{(3^2 1^2)}$.

Degenerate Whittaker models/coefficients, IV

This can be generalized to the orbit $((2k-1)^n 1^n)$:
one can choose a nice representative so that the stabilizer of this representative contains the image of

$$\mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{GL}_{2kn}, \qquad (g_1, g_2) \mapsto \mathrm{diag}(g_1, g_2, g_1, g_1, \cdots, g_1, g_1).$$

Degenerate Whittaker models/coefficients, V

The Whittaker model

This is attached to the orbit (n):
For example, if $G = GL_n$ and

$$N = \left\{ u = \begin{pmatrix} 1 & u_{12} & * & \cdots & * \\ & 1 & u_{23} & \cdots & * \\ & & 1 & \cdots & * \\ & & & & \vdots \\ & & & & 1 \end{pmatrix}. \right\}$$

A generic character is of the form

$$\psi_N(u) = \psi(u_{12} + u_{23} + \cdots + u_{n-1,n})$$

where ψ is a nontrivial additive character of $F\backslash\mathbb{A}$.

Degenerate Whittaker models/coefficients, VI

One can write this as

$$\psi_N(u) = \psi(\mathrm{tr}(f_{(n)} u))$$

where

$$f_{(n)} = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & & & \vdots \\ & & & 1 & 0 \\ & & & & 0 \end{pmatrix}$$

Consider

$$N_{(3^2)} = \left\{ u = \begin{pmatrix} I & X_1 & Y \\ & I & X_2 \\ & & I \end{pmatrix} \right\}$$

and

$$\psi_{(3^2)}(u) = \psi(\mathrm{tr}(X_1 + X_2)).$$

Equivalently,

$$\psi_{(3^2)}(u) = \psi(\mathrm{tr}(f_{(3^2)} u)).$$

Degenerate Whittaker models/coefficients, VII

Whittaker pair (S, φ)

- $(S, \varphi) \in \mathfrak{g} \times \mathfrak{g}^*$ such that S is rational semi-simple and $\mathrm{ad}_*(S)(\varphi) = -2\varphi$.
- Using the Killing form, $\varphi \leftrightarrow f \in \mathfrak{g}$, and f is nilpotent

Degenerate Whittaker model

- $\mathfrak{g} = \oplus_i \mathfrak{g}_i$ according to eigenvalues of S; assume that 1 is not an eigenvalue
- $\mathfrak{n} = \oplus_{i>1} \mathfrak{g}_i$ and $N = \exp(\mathfrak{n})$
- $\varphi|_{\mathfrak{n}}$ is a character of $\mathfrak{n}$ and hence a character ψ_N of N

Degenerate Whittaker models/coefficients, VIII

Degenerate Whittaker models/coefficients

For a representation π, locally we consider

$$\mathrm{Hom}_N(\pi, \psi_N).$$

Globally, for $\phi \in \pi$, we consider

$$\int_{N(F)\backslash N(\mathbb{A})} f(ug)\psi_N(u)\, du.$$

Nilpotent orbit attached to a representation

We say that the nilpotent orbit attached to a representation π is $\mathcal{O}$ if $\mathcal{O}$ is the maximal nilpotent orbit that supports a nonzero degenerate Whittaker model/coefficient.

Generalized Speh representations

Fix an integer n.

- τ: irreducible cuspidal automorphic representation of $\mathrm{GL}_k(\mathbb{A})$
- Let $\theta(n,\tau)$ be the unique irreducible quotient of

$$\tau|\cdot|^{(n-1)/2} \times \tau|\cdot|^{(n-3)/2} \times \cdots \times \tau|\cdot|^{-(n-1)/2}.$$

In other words,

$$\tau \in Irr_{gen}(\mathrm{GL}_k) \mapsto \theta(n,\tau) \in Irr(\mathrm{GL}_{kn}).$$

Key properties:

- the nilpotent orbit attached to $\theta(n,\tau)$ is (k^n).
- at every local place v, there is a unique model of degenerate type for $\theta(n,\tau)_v$.

The twisted doubling integrals, III

A Fourier coefficient in the orbit $((2k-1)^n 1^n)$

One can choose a nice pair in the orbit $((2k-1)^n 1^n)$ which gives (U, ψ_U) such that

$$\iota_k(G \times G) \subset \mathrm{Stab}(U, \psi_U).$$

(*This does not appear when* $k = 1$.)

Eisenstein series

- τ: irreducible cuspidal automorphic representation of $\mathrm{GL}_k(\mathbb{A})$
- $\theta(n, \tau)$: the generalized Speh representation of $\mathrm{GL}_{kn}(\mathbb{A})$
- $I(s, \theta(n, \tau)) = \mathrm{Ind}_{P(\mathbb{A})}^{G^{\square,k}(\mathbb{A})} \theta(n, \tau) \cdot \delta_P^s$
- $f^{(s)} \in I(s, \theta(n, \tau))$, one attaches an Eisenstein series

$$E(f^{(s)})(g) = \sum_{\gamma \in P(F) \backslash G^{\square,k}(F)} f^{(s)}(\gamma g).$$

The twisted doubling integrals, IV

- π: irreducible cuspidal representation of $G(\mathbb{A})$
- $\xi_1 \in \pi$ and $\xi_2 \in \pi^\vee$

The global integral

We define $Z(\xi_1 \boxtimes \xi_2, f^{(s)})$

$$\int_{G(F)\backslash G(\mathbb{A})\times G(F)\backslash G(\mathbb{A})} \xi_1(g_1)\xi_2(g_2)\cdot$$

$$\int_{U(F)\backslash U(\mathbb{A})} E(f^{(s)})(u \cdot \iota(g_1, g_2))\ \psi_U(u)\ dg_1\ dg_2.$$

The twisted doubling integrals, V

Unfolding

$$Z(\xi_1 \boxtimes \xi_2, f^{(s)}) = \text{Euler product...}$$

Unramified calculation

$$Z(\xi_1 \boxtimes \xi_2, f^{(s)}) \approx L(s, \pi \times \tau).$$

Quaternionic unitary groups, I

- D: quaternion algebra over F
- W is a free left D-module of rank n
- $\langle \ , \ \rangle$: a non-degenerate quadratic form on W
- $G = G(\mathcal{W})$
- G is an inner form of Sp_{2n} or O_{2n}.

Observations:

- One has to construct the generalized Speh representations on $\mathrm{GL}_{kn,D}(\mathbb{A})$ as the inducing data in the Eisenstein series. And verify expected properties.
- The other parts of twisted doubling integrals work without essential change.

How to construct θ?

Quaternionic unitary groups, II

Question

What can we say about the category $\mathrm{Rep}(\mathrm{GL}_{k,D})$?

Fact: Most irreducible representations of $\mathrm{GL}_{k,D}$ do not have unique models.

Example

- There is no nontrivial nilpotent elements in $D^\times$.
  (Nilpotent orbits of $\mathrm{GL}_{k,D}$ are classified by partitions of k.)
- So the only irreducible representations of $D^\times$ that have unique models are the one-dimensional representations.

Quaternionic unitary groups, III

Naive option

$$\tau \in \mathrm{Irr}(\mathrm{GL}_{k,D}) \mapsto \theta(n,\tau) \in \mathrm{Irr}(\mathrm{GL}_{kn,D}).$$

such that the nilpotent orbit attached to $\theta(n,\tau)$ is $(k^n)_D$.

Outcomes

- If this construction is possible, then we will obtain the tensor product L-function for $G \times \mathrm{GL}_{k,D}$.
- However, the orbit attached to $\theta(n,\tau)$ might not be correct.

Quaternionic unitary groups, IV

Another option

A construction

$$\tau \in \mathrm{Irr}_{gen}(\mathrm{GL}_k) \mapsto \theta_D(n, \tau) \in \mathrm{Irr}(\mathrm{GL}_{kn,D}).$$

It can be constructed by the following diagram:

$$\begin{tikzcd}
 & \mathrm{Irr}(\mathrm{GL}_{2kn}) \arrow[d, "|\mathrm{JL}|"] \\
\mathrm{Irr}_{gen}(\mathrm{GL}_k) \arrow[ur, "\theta(2n, -)"] \arrow[r, "\theta_D(n, -)"] & \mathrm{Irr}(\mathrm{GL}_{kn,D})
\end{tikzcd}$$

Here: $|\mathrm{JL}|$ is the Jacquet-Langlands correspondence in [Badulescu,2008] and [Badulescu-Renard, 2010], which is *local-to-global compatible*.

Quaternionic unitary groups, V

- Obtain L-funtion for $G \times \mathrm{GL}_{k,F}$; more natural when applying the Converse Theorem.
- The expected properties seems correct (at least at unramified places); but not easy to prove in general.

Quaternionic unitary groups, VI

A special instance

Let τ be an n-dimensional irreducible representation of $D^\times$ with $n > 1$. Let $\theta(2, \tau)$ be the unique irreducible subrepresentation of $\tau \times \tau\nu$. Here $\nu : D^\times \to \mathbb{C}^\times$ is the reduced norm. Then one expects

$$\dim \mathrm{Hom}_N(\theta(2, \tau), \psi_N) = 1.$$

Here $N = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \right\} \subset \mathrm{GL}_{2,D}$ and $\psi_N \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \psi(x)$ for a nontrivial additive character ψ of D.