Periods of cusp forms on $\text{GSp}_4$

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We present our recent progress on the following topics:

1. Automorphic analogue of Yoshida’s period relation for $\text{GSp}_4$.
2. Algebraicity of the symmetric sixth power $L$-functions for $\text{GL}_2$.
3. Algebraicity of critical values of the Rankin-Selberg $L$-functions for $\text{GSp}_4 \times \text{GL}_2$. 
Deligne’s conjecture for the spinor $L$-functions for $\text{GSp}_4$

- $\pi$: an irr. cusp. auto. rep. of $\text{GSp}_4(\mathbb{A})$ with trivial central character.
- We assume $\pi$ is globally generic and $\pi_{\infty}$ is a discrete series representation.

$$\pi_{\infty}|_{\text{Sp}_4(\mathbb{R})} = D(\lambda_1, \lambda_2) \oplus D(-\lambda_2, -\lambda_1),$$

where $D(\lambda_1, \lambda_2)$ is the discrete series representation of $\text{Sp}_4(\mathbb{R})$ with Blattner parameter $(\lambda_1, \lambda_2) \in \mathbb{Z}^2$ such that $1 - \lambda_1 \leq \lambda_2 \leq -1$.

- $\mathbb{Q}(\pi)$: the rationality field of $\pi$.
- $L(s, \pi)$: the spinor $L$-function of $\pi$.
- We assume $\pi$ is stable, that is, the functorial lift of $\pi$ to $\text{GL}_4(\mathbb{A})$ is cuspidal.
Deligne’s conjecture for the spinor $L$-functions for $\text{GSp}_4$

- $M$: the hypothetical motive attached to the spinor $L$-function $L(s, \pi)$.
- $M$ is a motive over $\mathbb{Q}$ with coefficients in $\mathbb{Q}(\pi)$ of rank 4 and of pure weight $w = \lambda_1 - \lambda_2 - 1$.
- $c^{\pm}(M) \in (\mathbb{Q}(\pi) \otimes \mathbb{C})^\times$: Deligne’s periods attached to $M$.
- Motivic $L$-function of $M$:
  \[ L(M, s) = \left( L^{(\infty)} \left( s - \frac{w}{2}, \pi^\sigma \right) \right)_{\sigma}, \]
  where $\sigma$ runs over a complete set of coset representatives of $\text{Aut}(\mathbb{C})/\text{Aut}(\mathbb{C}/\mathbb{Q}(\pi))$.

**Conjecture (Deligne (1977))**

Let $m \in \mathbb{Z}$ be a critical point for $M$. For any finite order character $\chi$ of $\mathbb{Q}^\times \setminus \mathbb{A}^\times$, we have

\[
\frac{L(M \otimes \chi, m)}{(2\pi\sqrt{-1})^{2m} \cdot G(\chi)^2 \cdot c(-1)^m \text{sgn}(\chi)(M)} \in \mathbb{Q}(\pi).
\]

Here $G(\chi)$ is the Gauss sum of $\chi$. 
We have the following automorphic analogue of Deligne’s conjecture.


There exist $c^\pm(\pi) \in \mathbb{C}^\times$ such that

$$
\left( \frac{L\left( \frac{1}{2} + m, \pi \times \chi \right)}{G(\chi)^2 \cdot c(-1)^{m\text{sgn}(\chi)}(\pi)} \right)^\sigma = \frac{L\left( \frac{1}{2} + m, \pi^\sigma \times \chi^\sigma \right)}{G(\chi^\sigma)^2 \cdot c(-1)^{m\text{sgn}(\chi)}(\pi^\sigma)}
$$

for all $\sigma \in \text{Aut}(\mathbb{C})$, all critical points $\frac{1}{2} + m \in \frac{1}{2} + \mathbb{Z}$ of $L(s, \pi)$, and any finite order character $\chi$ of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$.

**Remark**

The theorem is a special case of the results on the algebraicity of the critical values of the twisted standard $L$-functions of irr. regular algebraic, symplectic cusp. auto. rep. of $\text{GL}_{2n}$. 
Yoshida’s period relation

- $\pi^{\text{hol}}$: the unique irr. holo. cusp. auto. rep. of $\text{GSp}_4(\mathbb{A})$ such that
  \[ \pi^{\text{hol}}_f \simeq \pi_f. \]

- $f^{\text{hol}}$: a non-zero vector-valued holomorphic cusp form associated to $\pi^{\text{hol}}$ normalized so that its Fourier coefficients belong to $\mathbb{Q}(\pi)$.

- $c^{\pm}(\text{Sym}^2(M))$: Deligne’s periods attached to $\text{Sym}^2(M)$.

Theorem (Yoshida (2001))

We have

\[
\frac{c^+(\text{Sym}^2(M))}{(2\pi \sqrt{-1})^{6-3\lambda_1+3\lambda_2} \cdot c^+(M) \cdot c^-(M) \cdot (\|f^{\text{hol}}_\sigma\|)_\sigma}\in \mathbb{Q}(\pi).
\]
Deligne’s conjecture for the adjoint $L$-functions for $\text{GSp}_{4}$

- $L(s, \pi, \text{Ad})$: the adjoint $L$-function of $\pi$.

- Motivic $L$-function of $\text{Sym}^2(M)$:

$$L(\text{Sym}^2(M), s) = \left( L^{(\infty)}(s - w, \pi^\sigma, \text{Ad}) \right)_\sigma.$$ 

**Conjecture (Deligne (1977))**

Let $m \in \mathbb{Z}$ be a critical point for $M$. We have

$$\frac{L(\text{Sym}^2(M), m)}{(2\pi \sqrt{-1})^{d(-1)^m} \cdot c(-1)^m(\text{Sym}^2(M))} \in \mathbb{Q}(\pi).$$

Here $d^+ = 6$ and $d^- = 4$.

In particular, when $m$ is even, we have

$$\frac{L(\text{Sym}^2(M), m)}{(2\pi \sqrt{-1})^{6+6m-3\lambda_1+3\lambda_2} \cdot c^+(M) \cdot c^-(M) \cdot (\| f_{\text{hol}}^\sigma \|)_\sigma} \in \mathbb{Q}(\pi).$$
Main result

Following is our main result, which can be regarded as an automorphic analogue of Yoshida's period relation.

**Theorem (C.-)**

Assume that \( \lambda_1 + \lambda_2 \geq 4 \) and \( \lambda_2 \leq -5 \). For \( \sigma \in \text{Aut}(\mathbb{C}) \), we have

\[
\left( \frac{L(1, \pi, \text{Ad})}{\pi^3 \cdot c^+(\pi) \cdot c^-(\pi) \cdot \|f_{\text{hol}}\|} \right)^\sigma = \frac{L(1, \pi^\sigma, \text{Ad})}{\pi^3 \cdot c^+(\pi^\sigma) \cdot c^-(\pi^\sigma) \cdot \|f_{\text{hol}}^\sigma\|}.
\]

**Remark**

(1) The result holds for any irr. (tempered) stable cusp. auto. rep. \( \pi \) of \( \text{GSp}_4(\mathbb{A}) \) with trivial central character such that \( \pi_{\infty} \) is a discrete series representation.

(e.g. \( \pi = \pi_F \) with Hecke eigenform \( F \in S_{2-\lambda_2, \lambda_1+\lambda_2-2}(\text{Sp}_4(\mathbb{Z})) \).)

(2) In case \( \lambda_1 + \lambda_2 = 2 \), \( \pi_p \) is unramified for all primes \( p \), and there exists a quadratic character \( \chi \) of \( \mathbb{Q}^\times \backslash \mathbb{A}^\times \) with \( \chi_{\infty}(-1) = -1 \) such that

\[
L \left( \frac{1}{2}, \pi \right) L \left( \frac{1}{2}, \pi \times \chi \right) \neq 0.
\]

Then the theorem follows from the explicit refinement of Böcherer's conjecture proved by Furusawa–Morimoto.
Symmetric sixth power $L$-functions for $\GL_2$

- $\tau \subset A_0(\PGL_2(\mathbb{A}))$: non-dihedral with $\tau_\infty \simeq D(\kappa)$ for some $\kappa \geq 2$.
- $\pi \subset A_0(\GSp_4(\mathbb{A}))$: the automorphic descent of $\Sym^3 \tau$.

**Corollary**

Assume $\kappa \geq 6$. For $\sigma \in \Aut(\mathbb{C})$, we have

$$
\left( \frac{L(1, \tau, \Sym^6)}{\pi^6 \cdot \| f_\tau \|^3 \cdot \| F_\tau \|} \right)^\sigma = \frac{L(1, \tau^\sigma, \Sym^6)}{\pi^6 \cdot \| f_\sigma \|^3 \cdot \| F_\sigma \|}.
$$

Here

- $f_\tau$ is the normalized newform of $\tau$,
- $F_\tau$ is a non-zero vector-valued holomorphic cusp form associated to $\pi^{\text{hol}}$ normalized so that its Fourier coefficients belong to $\mathbb{Q}(\tau)$.

**Theorem (Morimoto)**

For $\sigma \in \Aut(\mathbb{C})$, we have

$$
\left( \frac{\| F_\tau \|}{\| f_\tau \|^3} \right)^\sigma = \pm \frac{\| F_\sigma \|}{\| f_\sigma \|^3}.
$$
Combining the corollary with Morimoto’s result, we obtain the following theorem.

**Theorem**

For $\sigma \in \text{Aut}(\mathbb{C})$, we have

$$
\left( \frac{L(1, \tau, \text{Sym}^6)}{\pi^6 \cdot \| f_\tau \|^6} \right)^\sigma = \pm \frac{L(1, \tau^\sigma, \text{Sym}^6)}{\pi^6 \cdot \| f_\sigma^\tau \|^6}.
$$

**Remark**

(1) Morimoto proved in a different way that
- the ratio is in $\overline{\mathbb{Q}}$,
- elliptic modular form $\rightarrow$ Hilbert modular form,
- all critical values,
- twisted symmetric fourth power $L$-function.

(2) When $\tau$ has level 1 and $F_\tau$ is a Hecke eigenform, we call $F_\tau$ the Kim–Ramakrishnan–Shahidi lift of $f_\tau$. Katsurada–Takemori conjectured that, after suitably normalized, a prime ideal dividing the ratio but not dividing $\Gamma(2\kappa)$ gives a congruence between $F_\tau$ and non K-R-S lift.
We show that there exist $\Omega^W_{\pm}(\pi) \in \mathbb{C}^\times$, call the Whittaker periods of $\pi$, satisfying the following assertions:

1. For $\sigma \in \text{Aut}(\mathbb{C})$, we have

$$
\left( \frac{L \left( \frac{1}{2} + m, \pi \right) L \left( \frac{\lambda_1 + \lambda_2 - 1}{2}, \pi \times \chi \right)}{\pi^{-2\lambda_1} \cdot G(\chi)^2 \cdot \Omega^W_+(\pi)} \right)^\sigma = \frac{L \left( \frac{1}{2} + m, \pi^\sigma \right) L \left( \frac{\lambda_1 + \lambda_2 - 1}{2}, \pi^\sigma \times \chi^\sigma \right)}{\pi^{-2\lambda_1} \cdot G(\chi^\sigma)^2 \cdot \Omega^W_+(\pi^\sigma)}
$$

for any finite order character $\chi$ of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$ and critical points $\frac{1}{2} + m$ such that $(-1)^m \frac{\lambda_1 + \lambda_2}{2} \chi_\infty(-1) = 1$.

2. For $\sigma \in \text{Aut}(\mathbb{C})$, we have

$$
\left( \frac{L \left( m, \pi \times \tau \right)}{\pi^{-\lambda_1 + \lambda_2} \cdot G(\chi_\tau)^2 \cdot \Omega^W_+(\pi) \cdot \|f_\tau\|} \right)^\sigma = \frac{L \left( m, \pi^\sigma \times \tau^\sigma \right)}{\pi^{-\lambda_1 + \lambda_2} \cdot G(\chi_\tau^\sigma)^2 \cdot \Omega^W_+(\pi^\sigma) \cdot \|f_\tau^\sigma\|}
$$

for any irr. cusp. auto. rep. $\tau$ of $\text{GL}_2(\mathbb{A})$ satisfying:

(i) $\omega_\tau = \chi_\tau \mid r_\mathbb{A}$ for some finite order character $\chi_\tau$ of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$ and $r \in \mathbb{Z}$,

(ii) $\tau_\infty \otimes \mid r_\mathbb{R}^{-r/2} \simeq D(\ell)$ for some $\lambda_1 + \lambda_2 + 1 \leq \ell \leq \lambda_1$ with $\ell \equiv r \pmod{2}$,

and any critical points $m \in \mathbb{Z}$ with $m > -\frac{r}{2}$. 

Proof of the main result

We show that there exist $\Omega^W_{\pm}(\pi) \in \mathbb{C}^\times$, call the Whittaker periods of $\pi$, satisfying the following assertions:

1. For $\sigma \in \text{Aut}(\mathbb{C})$, we have

$$
\left( \frac{L \left( \frac{1}{2} + m, \pi \right) L \left( \frac{\lambda_1 + \lambda_2 - 1}{2}, \pi \times \chi \right)}{\pi^{-2\lambda_1} \cdot G(\chi)^2 \cdot \Omega^W_+(\pi)} \right)^\sigma = \frac{L \left( \frac{1}{2} + m, \pi^\sigma \right) L \left( \frac{\lambda_1 + \lambda_2 - 1}{2}, \pi^\sigma \times \chi^\sigma \right)}{\pi^{-2\lambda_1} \cdot G(\chi^\sigma)^2 \cdot \Omega^W_+(\pi^\sigma)}
$$

for any finite order character $\chi$ of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$ and critical points $\frac{1}{2} + m$ such that $(-1)^m \frac{\lambda_1 + \lambda_2}{2} \chi_\infty(-1) = 1$.

2. For $\sigma \in \text{Aut}(\mathbb{C})$, we have

$$
\left( \frac{L \left( m, \pi \times \tau \right)}{\pi^{-\lambda_1 + \lambda_2} \cdot G(\chi_\tau)^2 \cdot \Omega^W_+(\pi) \cdot \|f_\tau\|} \right)^\sigma = \frac{L \left( m, \pi^\sigma \times \tau^\sigma \right)}{\pi^{-\lambda_1 + \lambda_2} \cdot G(\chi_\tau^\sigma)^2 \cdot \Omega^W_+(\pi^\sigma) \cdot \|f_\tau^\sigma\|}
$$

for any irr. cusp. auto. rep. $\tau$ of $\text{GL}_2(\mathbb{A})$ satisfying:

(i) $\omega_\tau = \chi_\tau \mid r_\mathbb{A}$ for some finite order character $\chi_\tau$ of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$ and $r \in \mathbb{Z}$,

(ii) $\tau_\infty \otimes \mid r_\mathbb{R}^{-r/2} \simeq D(\ell)$ for some $\lambda_1 + \lambda_2 + 1 \leq \ell \leq \lambda_1$ with $\ell \equiv r \pmod{2}$,

and any critical points $m \in \mathbb{Z}$ with $m > -\frac{r}{2}$.
Proof of the main result

(3) For $\sigma \in \text{Aut}(\mathbb{C})$, we have

$$
\left( \frac{\| f_\pi \|_{\Omega^W_+ (\pi) \cdot \Omega^W_- (\pi)}}{\Omega^W_+ (\pi) \cdot \Omega^W_- (\pi)} \right)^\sigma = \frac{\| f_{\pi^\sigma} \|_{\Omega^W_+ (\pi^\sigma) \cdot \Omega^W_- (\pi^\sigma)}}{\Omega^W_+ (\pi^\sigma) \cdot \Omega^W_- (\pi^\sigma)}.
$$

Here $f_\pi$ is the normalized newform of $\pi$ (to be explained later).

(1) + results of Januszewski, Jiang–Sun–Tian: if $\lambda_1 + \lambda_2 \geq 4$, then

$$
\left( \frac{\Omega^W_+ (\pi)}{\pi^{2\lambda_1} \cdot c^+ (\pi) \cdot c^- (\pi)} \right)^\sigma = \frac{\Omega^W_+ (\pi^\sigma)}{\pi^{2\lambda_1} \cdot c^+ (\pi^\sigma) \cdot c^- (\pi^\sigma)}.
$$

(2) + results of Furusawa, Böcherer–Heim, Pitale–Schmidt, Saha, Morimoto: if $\lambda_2 \leq -5$, then

$$
\left( \frac{\Omega^W_- (\pi)}{\pi^{4+\lambda_1 - \lambda_2} \cdot \| f_{\text{hol}} \|} \right)^\sigma = \frac{\Omega^W_- (\pi^\sigma)}{\pi^{4+\lambda_1 - \lambda_2} \cdot \| f_{\text{hol}}^\sigma \|}.
$$

Therefore, by (3) we have

$$
\left( \frac{\| f_\pi \|_{\pi^{4+3\lambda_1 - \lambda_2} \cdot c^+ (\pi) \cdot c^- (\pi) \cdot \| f_{\text{hol}} \|}}{\pi^{4+3\lambda_1 - \lambda_2} \cdot c^+ (\pi) \cdot c^- (\pi) \cdot \| f_{\text{hol}} \|} \right)^\sigma = \frac{\| f_{\pi^\sigma} \|_{\pi^{4+3\lambda_1 - \lambda_2} \cdot c^+ (\pi^\sigma) \cdot c^- (\pi^\sigma) \cdot \| f_{\text{hol}}^\sigma \|}}{\pi^{4+3\lambda_1 - \lambda_2} \cdot c^+ (\pi^\sigma) \cdot c^- (\pi^\sigma) \cdot \| f_{\text{hol}}^\sigma \|}.
$$
Our main theorem then follows from the following result:

- For $\sigma \in \text{Aut}(\mathbb{C})$, we have
  \[
  \left( \frac{L(1, \pi, \text{Ad})}{\pi^{1-3\lambda_1+\lambda_2} \cdot \|f_\pi\|} \right)^\sigma = \frac{L(1, \pi^\sigma, \text{Ad})}{\pi^{1-3\lambda_1+\lambda_2} \cdot \|f_{\pi^\sigma}\|}.
  \]

C.—Ichino (2019): We computed the explicit value for the ratio when $\pi$ has square-free paramodular conductor.

In the rest of this talk, we

- sketch the construction of the Whittaker periods $\Omega_\pm^W(\pi)$,
- sketch the proof of the corresponding algebraicity results for critical $L$-values.
For \((k_1, k_2) \in \mathbb{Z}^2\) with \(k_1 \geq k_2\), let \((\rho_{(k_1,k_2)}, V_{(k_1,k_2)})\) be the irr. alg. rep. of \(U(2)\) defined by
\[
\rho_{(k_1,k_2)} = \text{Sym}^{k_1-k_2} \otimes \text{det}^{k_2},
\]
\[
V_{(k_1,k_2)} = \langle X^i Y^{k_1-k_2-i} \mid 0 \leq i \leq k_1 - k_2 \rangle_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}.
\]

Maximal unipotent subgroup of \(GSp_4\):
\[
U = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \end{pmatrix} \in GSp_4 \right\}.
\]

\(\psi_U : U(\mathbb{Q}) \backslash U(\mathbb{A}) \longrightarrow \mathbb{C}^\times\) defined by
\[
\psi_U \begin{pmatrix} 1 & x & * & * \\ 0 & 1 & * & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x & 1 \end{pmatrix} = \psi(-x - y),
\]
where \(\psi : \mathbb{Q} \backslash \mathbb{A} \longrightarrow \mathbb{C}^\times\) so that \(\psi_\infty(x) = e^{2\pi \sqrt{-1} x}\).
For $\varphi \in \pi$, define the global Whittaker function

$$W_{\varphi, \psi_U}(g) = \int_{U(\mathbb{Q}) \setminus U(\mathbb{A})} \varphi(ug)\overline{\psi_U(u)} \, du.$$ 

$\mathcal{W}(\pi_v, \psi_{U,v})$: the space of Whittaker functions of $\pi_v$ with respect to $\psi_{U,v}$.

$\mathcal{W}(\pi_f, \psi_{U,f}) = \bigotimes'_p \mathcal{W}(\pi_p, \psi_{U,p})$.

For $\sigma \in \text{Aut} (\mathbb{C})$, define the $\sigma$-linear isomorphism of $GSp_4(\mathbb{A}_f)$-modules:

$$t_\sigma : \mathcal{W}(\pi_f, \psi_{U,f}) \longrightarrow \mathcal{W}(\pi_f^\sigma, \psi_{U,f}),$$

$$t_\sigma W(g) = \sigma \left( W \left( \text{diag}(u^{-2}, u^{-1}, u, 1)g \right) \right).$$

Here $\sigma|_{\mathbb{Q}^{ab}} = \text{rec}(a \cdot u)$ with $a \cdot u \in \mathbb{R}_{>0} \cdot \hat{\mathbb{Z}}^\times$ and $\text{rec} : \mathbb{Q}^\times \setminus \mathbb{A}^\times \longrightarrow \text{Gal} (\mathbb{Q}^{ab}/\mathbb{Q})$ is the geometrically normalized reciprocity map.
Recall

$$\pi_{\infty}|_{\text{Sp}_4(\mathbb{R})} = D(\lambda_1, \lambda_2) \oplus D(-\lambda_2, -\lambda_1),$$

for some \((\lambda_1, \lambda_2) \in \mathbb{Z}^2\) such that \(1 - \lambda_1 \leq \lambda_2 \leq -1\).

Write

$$\pi^+ = (\pi \otimes_{\mathbb{C}} V(\lambda_1, \lambda_2))^{U(2)}, \quad \pi^- = (\pi \otimes_{\mathbb{C}} V(-\lambda_2, -\lambda_1))^{U(2)}.$$

Note that \(\pi^+ \simeq \pi^- \simeq \pi_f\) as \(\text{GSp}_4(\mathbb{A}_f)\)-modules.

For \(f \in \pi^+, \ h \in \pi^-,\) we have

$$f = \sum_{i=0}^{\lambda_1 - \lambda_2} (-1)^i \binom{\lambda_1 - \lambda_2}{i} \cdot P_i^+(f) \otimes X^{\lambda_1 - \lambda_2 - i} Y^i,$$

$$h = \sum_{i=0}^{\lambda_1 - \lambda_2} \binom{\lambda_1 - \lambda_2}{i} \cdot P_i^-(h) \otimes X^i Y^{\lambda_1 - \lambda_2 - i}$$

for some uniquely determined \(P_i^+(f), P_i^-(h) \in \pi\) for \(0 \leq i \leq \lambda_1 - \lambda_2\).
Let $0 \leq i \leq \lambda_1 - \lambda_2$.

- $W_i \in \mathcal{W}(\pi_\infty, \psi_{U,\infty})$: in the minimal $U(2)$-type of $D(-\lambda_2, -\lambda_1)$ with weight $(-\lambda_1 + i, -\lambda_2 - i)$ normalized so that (following T. Moriyama)

$$W_i(1) = (2\sqrt{-1})^{3\lambda_1 + \lambda_2 - i}\pi^{-\frac{1}{2}}e^{-2\pi}$$

$$\times \int_{c_1 - \sqrt{-1}\infty}^{c_1 + \sqrt{-1}\infty} \frac{ds_1}{2\pi\sqrt{-1}} \int_{c_2 - \sqrt{-1}\infty}^{c_2 + \sqrt{-1}\infty} \frac{ds_2}{2\pi\sqrt{-1}} (4\pi^3)^{-s_1 + \lambda_1 + 1 - i} (4\pi)^{-\frac{s_2 + \lambda_2 + i}{2}}$$

$$\times \Gamma\left(\frac{s_1 + s_2 - 2\lambda_2 + 1}{2}\right) \Gamma\left(\frac{s_1 + s_2 + 1}{2}\right) \Gamma\left(\frac{s_1}{2}\right) \Gamma\left(\frac{-s_2}{2}\right) \Gamma(s_1),$$

where $c_1, c_2 \in \mathbb{R}$ satisfy $c_1 + c_2 + 1 > 0$ and $c_1 > 0 > c_2$.

- $\overline{W}_i \in \mathcal{W}(\pi_\infty, \overline{\psi}_{U,\infty})$: in the minimal $U(2)$-type of $D(\lambda_1, \lambda_2)$ with weight $(\lambda_1 - i, \lambda_2 + i)$. 

Rational structures via the Whittaker models
Define $\text{GSp}_4(\mathbb{A}_f)$-module isomorphisms

$$\pi^+ \longrightarrow \mathcal{W}(\pi_f, \psi_{U,f}), \quad f \longmapsto W_f^+$$

$$\pi^- \longrightarrow \mathcal{W}(\pi_f, \overline{\psi}_{U,f}), \quad h \longmapsto W_h^-$$

by

$$W_{P_i^+(f),\psi_U} = W_i \cdot W_f^+, \quad W_{P_i^-(h),\overline{\psi}_U} = \overline{W_i} \cdot W_h^-$$

for $0 \leq i \leq \lambda_1 - \lambda_2$.

Define the $\sigma$-linear isomorphisms of $\text{GSp}_4(\mathbb{A}_f)$-modules

$$\pi^+ \longrightarrow (\pi^\sigma)^+, \quad f \longmapsto f^\sigma$$

$$\pi^- \longrightarrow (\pi^\sigma)^-, \quad h \longmapsto h^\sigma$$

by

$$W_{f^\sigma}^+ = t_\sigma W_f^+, \quad W_{h^\sigma}^- = t_\sigma W_h^-.$$

$f_\pi \in \pi^+$: the normalized newform of $\pi$ defined so that $W_{f_\pi}^+ \in \mathcal{W}(\pi_f, \psi_{U,f})$ is the paramodular newform with $W_{f_\pi}^+(1) = 1$. It is clear that $f_\pi^\sigma = f_{\pi \sigma}$ for $\sigma \in \text{Aut}(\mathbb{C})$. 

Rational structures via the cohomology

- Put $K_\infty = \mathbb{R}^\times \cdot U(2) \subset \text{GSp}_4(\mathbb{R})$ and (cf. Oshima’s and Horinaga’s talk)
  \[ p^- = \left\{ \left( \begin{array}{cc} \sqrt{-1}A & A \\ A & \sqrt{-1}A \end{array} \right) \bigg| A = tA \in M_2(\mathbb{Q}) \right\} \otimes_\mathbb{Q} \mathbb{C} \subset \text{Lie}(\text{GSp}_4(\mathbb{R}))_\mathbb{C}. \]

- Let $(\rho, V_\rho)$ be an irr. alg. rep. of $K_\infty$. Consider the complexes with respect to the Lie algebra differential operator:
  \[
  \begin{align*}
  C^q_{\text{sia}, \rho} &= \left( C^\infty_{\text{sia}}(\text{GSp}_4(\mathbb{Q}) \backslash \text{GSp}_4(\mathbb{A})) \otimes_\mathbb{C} \bigwedge^q (p^-)^* \otimes_\mathbb{C} V_\rho \right)^{K_\infty}, \\
  C^q_{\text{rda}, \rho} &= \left( C^\infty_{\text{rda}}(\text{GSp}_4(\mathbb{Q}) \backslash \text{GSp}_4(\mathbb{A})) \otimes_\mathbb{C} \bigwedge^q (p^-)^* \otimes_\mathbb{C} V_\rho \right)^{K_\infty}
  \end{align*}
  \]
  for $q \geq 0$.

- $H^q(\mathcal{V}^\text{can}_\rho)$ and $H^q(\mathcal{V}^\text{sub}_\rho)$: the $q$-th cohomology groups with respect to the complexes $C^*_{\text{sia}, \rho}$ and $C^*_{\text{rda}, \rho}$, respectively.

- $H^q_1(\mathcal{V}_\rho)$: the image of the morphism $H^q(\mathcal{V}^\text{sub}_\rho) \longrightarrow H^q(\mathcal{V}^\text{can}_\rho)$ induced by the inclusion $C^*_{\text{rda}, \rho} \longrightarrow C^*_{\text{sia}, \rho}$. 

Rational structures via the cohomology

**Theorem (Harris, Milne)**

1. \( H^q(\mathcal{V}_\rho^{\text{can}}) \) and \( H^q(\mathcal{V}_\rho^{\text{sub}}) \) are admissible \( \text{GSp}_4(\mathbb{A}_f) \)-modules and have canonical rational structures over \( \mathbb{Q} \).

2. \( H^q_i(\mathcal{V}_\rho) \) is semisimple.

3. For \( \sigma \in \text{Aut}(\mathbb{C}) \), conjugation by \( \sigma \) induces natural \( \sigma \)-linear \( \text{GSp}_4(\mathbb{A}_f) \)-module isomorphism:

\[
T_\sigma : H^q(\mathcal{V}_\rho^{\text{can}}) \longrightarrow H^q(\mathcal{V}_\rho^{\text{can}}).
\]

Similar assertion holds for \( H^q(\mathcal{V}_\rho^{\text{sub}}) \) and \( H^q_i(\mathcal{V}_\rho) \).

4. We have a natural injective homomorphism of \( \text{GSp}_4(\mathbb{A}_f) \)-modules

\[
\text{cl} : \left( \mathcal{A}_0(\text{GSp}_4(\mathbb{A}), \rho) \otimes_{\mathbb{C}} \bigwedge^q (\mathfrak{p}^-)^* \otimes_{\mathbb{C}} \mathcal{V}_\rho \right)^{K_\infty} \longrightarrow H^q_i(\mathcal{V}_\rho)
\]

for each \( q \in \mathbb{Z}_{\geq 0} \). Here \( \mathcal{A}_0(\text{GSp}_4(\mathbb{A}), \rho) \) is the space of cusp forms on \( \text{GSp}_4(\mathbb{A}) \) which are eigenfunctions of the Casimir operator of \( \text{GSp}_4(\mathbb{R}) \) with certain eigenvalue depending on \( \rho \).
We apply the results of Harris to \((\rho, V_\rho)\) equal to
\[
(\rho^+, V^+) := (\rho(\lambda_1-3, \lambda_2-1), V(\lambda_1-3, \lambda_2-1)),
\]
\[
(\rho^-, V^-) := (\rho(-\lambda_2-2, -\lambda_1), V(-\lambda_2-2, -\lambda_1)).
\]

In these two cases, \(\pi\) occurs in \(A_0(GSp_4(\mathbb{A}), \rho)\).

**Proposition**

The homomorphisms \(cl\) induce isomorphisms of \(GSp_4(\mathbb{A}_f)\)-modules:

\[
cl : \pi \otimes \mathbb{C} \bigwedge^2 (p^-)^* \otimes \mathbb{C} V^+ \xrightarrow{K_\infty} H^2_! (V_{\rho^+})[\pi_f] \cong \pi_f,
\]

\[
cl : \pi \otimes \mathbb{C} \bigwedge^1 (p^-)^* \otimes \mathbb{C} V^- \xrightarrow{K_\infty} H^1_! (V_{\rho^-})[\pi_f] \cong \pi_f.
\]

**Remark**

We use Arthur’s multiplicity formula to deduce that
\[
A_0(GSp_4(\mathbb{A}))[\pi_f] = \pi \oplus \pi^{\text{hol}}.
\]
Whittaker periods

Fix \( \mathbb{Q} \)-rational injective \( K_{\infty} \)-equivariant homomorphisms:

\[
V_{(\lambda_1, \lambda_2)} \longrightarrow \bigwedge^2 (p^-)^* \otimes_{\mathbb{C}} V^+ \Longrightarrow \text{cl} : \pi^+ \longrightarrow H^2(\mathcal{V}_{\rho^+})[\pi_f],
\]

\[
V_{(-\lambda_2, -\lambda_1)} \longrightarrow \bigwedge^1 (p^-)^* \otimes_{\mathbb{C}} V^- \Longrightarrow \text{cl} : \pi^- \longrightarrow H^1(\mathcal{V}_{\rho^-})[\pi_f].
\]

Note that the \( \mathbb{Q} \)-rational embeddings are unique up to homotheties over \( \mathbb{Q}^\times \).

**Lemma**

*There exist* \( \Omega^W_+ (\pi) \), \( \Omega^W_- (\pi) \) \( \in \mathbb{C}^\times \), *unique up to* \( \mathbb{Q}(\pi)^\times \), *such that*

\[
\frac{\text{cl} \left( \pi^+, \text{Aut} (\mathbb{C}/\mathbb{Q}(\pi)) \right)}{\Omega^W_+ (\pi)} = H^2(\mathcal{V}_{\rho^+})[\pi_f]^{\text{Aut} (\mathbb{C}/\mathbb{Q}(\pi))},
\]

\[
\frac{\text{cl} \left( \pi^-, \text{Aut} (\mathbb{C}/\mathbb{Q}(\pi)) \right)}{\Omega^W_- (\pi)} = H^1(\mathcal{V}_{\rho^-})[\pi_f]^{\text{Aut} (\mathbb{C}/\mathbb{Q}(\pi))}.
\]

We call \( \Omega^W_\pm (\pi) \) the **Whittaker periods** of \( \pi \). By the uniqueness up to \( \mathbb{Q}(\pi)^\times \), we can normalize the Whittaker periods so that

\[
T_\sigma \left( \frac{\text{cl}(f)}{\Omega^W_\varepsilon (\pi)} \right) = \frac{\text{cl}(f^\sigma)}{\Omega^W_\varepsilon (\pi^\sigma)}, \quad f \in \pi^\varepsilon.
\]
Algebraicity of critical $L$-values for $\text{GSp}_4 \times \text{GL}_2$

**Theorem (C.-, in progress)**

1. **For** $\sigma \in \text{Aut}(\mathbb{C})$, we have
   \[
   \left( \frac{L \left( \frac{1}{2} + m, \pi \right) L \left( \frac{\lambda_1 + \lambda_2 - 1}{2}, \pi \times \chi \right)}{\pi^{-2\lambda_1} \cdot G(\chi)^2 \cdot \Omega^W_+(\pi)} \right)^{\sigma} = \frac{L \left( \frac{1}{2} + m, \pi^{\sigma} \right) L \left( \frac{\lambda_1 + \lambda_2 - 1}{2}, \pi^{\sigma} \times \chi^{\sigma} \right)}{\pi^{-2\lambda_1} \cdot G(\chi^{\sigma})^2 \cdot \Omega^W_+(\pi^{\sigma})}
   \]
   for any finite order character $\chi$ of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$ and critical points $\frac{1}{2} + m$ such that $(-1)^{m + \frac{\lambda_1 + \lambda_2}{2}} \chi_\infty(-1) = 1$.

2. **For** $\sigma \in \text{Aut}(\mathbb{C})$, we have
   \[
   \left( \frac{L \left( m, \pi \times \tau \right)}{\pi^{-\lambda_1 + \lambda_2} \cdot G(\chi_{\tau})^2 \cdot \Omega^W_+(\pi) \cdot \|f_\tau\|} \right)^{\sigma} = \frac{L \left( m, \pi^{\sigma} \times \tau^{\sigma} \right)}{\pi^{-\lambda_1 + \lambda_2} \cdot G(\chi_{\tau}^{\sigma})^2 \cdot \Omega^W_+(\pi^{\sigma}) \cdot \|f_{\tau^{\sigma}}\|}
   \]
   for any irr. cusp. auto. rep. $\tau$ of $\text{GL}_2(\mathbb{A})$ satisfying:
   
   (i) $\omega_\tau = \chi_{\tau} | r_\mathbb{A}$ for some finite order character $\chi_{\tau}$ of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$ and $r \in \mathbb{Z}$,
   (ii) $\tau_\infty \otimes | -r/2 \simeq D(\ell)$ for some $\lambda_1 + \lambda_2 + 1 \leq \ell \leq \lambda_1$ with $\ell \equiv r \pmod{2}$,
   
   and any critical points $m \in \mathbb{Z}$ with $m > -\frac{r}{2}$.
We sketch the proof of (2). Write $\omega = \omega_\tau$ and $\chi = \chi_\tau$.

- Let $G' = \{(g_1, g_2) \in GL_2 \times GL_2 \mid \det(g_1) = \det(g_2)\}$ and we regard it as a subgroup of $GSp_4$ by the embedding

$$G' \rightarrow GSp_4, \quad \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) \mapsto \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix}.$$

Ingredients:

(1) Integral representation of $L(s, \pi \times \tau)$.

(2) Cohomological interpretation of the global integral.

(3) Local zeta integrals:
   - Explicit calculation of the archimedean local zeta integral.
   - Galois equivariance property of the $p$-adic local zeta integral.
(1) Integral representation of $L(s, \pi \times \tau)$ (Piatetski-Shapiro–Soudry):

- For $\varphi_1 \in \pi$, $\varphi_2 \in \tau$, and $f_s \in \text{Ind}_{B(\mathbb{A})}^{\text{GL}_2(\mathbb{A})}(\varphi_1(g)E(g; f_s)\varphi_2(g_2) \, dg

\begin{align*}
Z(\varphi_1, \varphi_2, E(f_s)) &:= \int_{\mathbb{A} \times G'(\mathbb{Q}) \backslash G'(\mathbb{A})} \varphi_1(g)E(g_1; f_s)\varphi_2(g_2) \, dg \\
&= \int_{\mathbb{A} \times N'(\mathbb{A}) \backslash G'(\mathbb{A})} W_{\varphi_1, \psi}(g)f_s(g_1)W_{\varphi_2, \psi}(g_2) \, dg.
\end{align*}

Here $dg$ is the Tamagawa measure on $\mathbb{A} \times \backslash G'(\mathbb{A})$.

- For $\kappa \geq 1$ and $\Phi \in S(\mathbb{A}_f^2)$, let $E_{s}^{[\kappa]}(\omega, \Phi) = E(f_{\omega, \Phi[\kappa] \otimes \Phi, s})$, where

\begin{align*}
f_{\omega, \Phi[\kappa] \otimes \Phi, s}(g) &= |\det(g)|_{\mathbb{A}}^{s} \int_{\mathbb{A} \times (0, t)g(0, t)} (\Phi[\kappa] \otimes \Phi)((0, t)g) \omega(t)|t|_{\mathbb{A}}^{2s} d^\times t
\end{align*}

and

\begin{align*}
\Phi[\kappa](x, y) &= 2^{-\kappa}(x + \sqrt{-1}y)^{\kappa} e^{-\pi(x^2+y^2)}.
\end{align*}

Then $E_{s}^{[\kappa]}(\omega, \Phi)|_{s=\frac{\kappa-r}{2}}$ is a holo. auto. form of weight $\kappa$. 

(2) Cohomological interpretation of the global integral.

- Let \( m \in \mathbb{Z} \) with \( -\frac{r}{2} < m \leq \frac{\ell - \lambda_1 - \lambda_2 - r}{2} \) (right-half critical points).

\[
H^1_* (\mathcal{V}_{\rho -}) \otimes_{C} H^1_* (\mathcal{V}_{(\ell-2; -r)}) \otimes_{C} H^0 (\mathcal{V}^{\text{can}}_{(\lambda_1 + \lambda_2 - \ell; r)}) \rightarrow \mathbb{C},
\]

\[
[f_1] \otimes [f_2] \otimes [f_3] \mapsto Z \left( P_{\ell - \lambda_2}^{-} (f_1), f_2, X^{\frac{\ell - \lambda_1 - \lambda_2 - r}{2} - m} \cdot f_3 \right)
\]

is a well-defined \( \mathbb{Q} \)-rational trilinear form. Here

- \( \mathcal{V}_{(\kappa; r)} \): the automorphic line bundle on \( \mathcal{M}_{\text{GL}_2} \) associated to the character

\[
a \cdot e^{\sqrt{-1} \theta} \mapsto a^r \cdot e^{\kappa \sqrt{-1} \theta}
\]

of \( \mathbb{R}^* \cdot U(1) \).

- \( X_+ = -\frac{1}{4} \begin{pmatrix} \sqrt{-1} & -1 \\ -1 & -\sqrt{-1} \end{pmatrix} \in \text{Lie}(\text{GL}_2(\mathbb{R}))_C \) is the weight raising differential operator.
For \( \varphi_1 \in \pi^- \), \( \varphi_2 \in \tau \) with weight \(-\ell\), and \( \Phi \in S(\mathbb{A}_f^2) \), we have

\[
T_\sigma \left( \frac{[\varphi_1]}{\Omega_\pi^W} \right) = \frac{[\varphi_1^\sigma]}{\Omega_\pi^W} \in H^1_! \left( \mathcal{V}_{\rho-} \right)[\pi^\sigma],
\]

\[
T_\sigma \left( \frac{[\varphi_2]}{\|f_\tau\|(2\pi \sqrt{-1})^{\frac{\ell-r}{2}}} \right) = \frac{[\varphi_2^\sigma]}{\|f_\tau\|(2\pi \sqrt{-1})^{\frac{\ell-r}{2}}} \in H^1_! \left( \mathcal{V}_{(\ell-2; -r)} \right)[\tau^\sigma],
\]

\[
T_\sigma \left( \frac{[E_s^{2m+r}](\omega, \Phi)_{s=m}}{G(\chi)(2\pi \sqrt{-1})^{-m}} \right) = \frac{[E_s^{2m+r}](\omega^\sigma, \Phi^\sigma)_{s=m}}{G(\chi^\sigma)(2\pi \sqrt{-1})^{-m}} \in H^0_! \left( \mathcal{V}_{\text{can}}^{(\lambda_1+\lambda_2-\ell; r)} \right).
\]

We conclude that

\[
Z \left( \frac{P_{\ell-\lambda_2}(\varphi_1), \varphi_2, X_{\frac{\ell-\lambda_1-\lambda_2-r}{2}}^{-m} \cdot E_s^{2m+r}(\omega, \Phi)_{s=m}}{\Omega_\pi^W \cdot \|f_\tau\|(2\pi \sqrt{-1})^{\frac{\ell-r}{2}} \cdot G(\chi)(2\pi \sqrt{-1})^{-m}} \right) =
\]

\[
Z \left( \frac{P_{\ell-\lambda_2}(\varphi_1^\sigma), \varphi_2^\sigma, X_{\frac{\ell-\lambda_1-\lambda_2-r}{2}}^{-m} \cdot E_s^{2m+r}(\omega^\sigma, \Phi^\sigma)_{s=m}}{\Omega_\pi^W \cdot \|f_\tau\|(2\pi \sqrt{-1})^{\frac{\ell-r}{2}} \cdot G(\chi^\sigma)(2\pi \sqrt{-1})^{-m}} \right).
\]
(3) Local zeta integrals:

- Let $\overline{W}_{\ell - \lambda_2} \in \mathcal{W}(\pi_\infty, \overline{\psi}_U, \infty)$: normalized with weight $(\lambda_1 + \lambda_2 - \ell, \ell)$,
- $W'_\ell \in \mathcal{W}(\tau_\infty, \overline{\psi}_\infty)$: weight $-\ell$ with $W'_\ell(1) = e^{-2\pi}$,
- $f_{\omega_\infty, \phi[\ell - \lambda_1 - \lambda_2], s}$: normalized with weight $\ell - \lambda_1 - \lambda_2$.

We have

$$Z_\infty(\overline{W}_{\ell - \lambda_2}, W'_\ell, f_{\omega_\infty, \phi[\ell - \lambda_1 - \lambda_2], s})$$

$$\in (\sqrt{-1})^{\frac{\lambda_1 + \lambda_2}{2}} \cdot \pi^{\frac{3\lambda_1 - \lambda_2}{2}} \cdot L(s, \pi_\infty \times \tau_\infty) \cdot \mathbb{Q}^\times.$$

- Let $p$ be a prime. Let $W_1 \in \mathcal{W}(\pi_p, \overline{\psi}_U, p)$, $W_2 \in \mathcal{W}(\tau_p, \overline{\psi}_p)$, and $\Phi \in S(\mathbb{Q}_p^2)$. For $\sigma \in \text{Aut}(\mathbb{C})$, we have

$$\left( Z_p(W_1, W_2, f_{\omega_p, \phi, s}) \right)^\sigma = Z_p(t_\sigma W_1, t_\sigma W_2, f_{\omega_\sigma, \phi, s}, s)$$

$$\varepsilon(0, \chi_p, \psi_p)$$

$$L(s, \pi_p \times \tau_p)^\sigma = L(s, \pi_\sigma^p \times \tau_\sigma^p).$$

The assertion then follows from (1)-(3) with good choice of datum

$$(W_1, W_2, \Phi) \in \mathcal{W}(\pi_f, \overline{\psi}_{U, f}) \times \mathcal{W}(\tau_f, \overline{\psi}_f) \times S(\mathbb{A}_f^2).$$
The end.
Thank you for your attention.