Periods of cusp forms on GSp_4

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We present our recent progress on the following topics:

(1) Automorphic analogue of Yoshida's period relation for GSp_4.

(2) Algebraicity of the symmetric sixth power L-functions for GL_2.

(3) Algebraicity of critical values of the Rankin-Selberg L-functions for $\mathrm{GSp}_4 \times \mathrm{GL}_2$.

- π: an irr. cusp. auto. rep. of $\mathrm{GSp}_4(\mathbb{A})$ with trivial central character.
- We assume π is globally generic and π_∞ is a discrete series representation.

$$\pi_\infty|_{\mathrm{Sp}_4(\mathbb{R})} = D_{(\lambda_1,\lambda_2)} \oplus D_{(-\lambda_2,-\lambda_1)},$$

  where $D_{(\lambda_1,\lambda_2)}$ is the discrete series representation of $\mathrm{Sp}_4(\mathbb{R})$ with Blattner parameter $(\lambda_1,\lambda_2) \in \mathbb{Z}^2$ such that $1-\lambda_1 \leq \lambda_2 \leq -1$.
- $\mathbb{Q}(\pi)$: the rationality field of π.
- $L(s,\pi)$: the spinor L-function of π.
- We assume π is stable, that is, the functorial lift of π to $\mathrm{GL}_4(\mathbb{A})$ is cuspidal.

Deligne's conjecture for the spinor L-functions for GSp_4

- M: the hypothetical motive attached to the spinor L-function $L(s,\pi)$.
- M is a motive over $\mathbb{Q}$ with coefficients in $\mathbb{Q}(\pi)$ of rank 4 and of pure weight $w = \lambda_1 - \lambda_2 - 1$.
- $c^{\pm}(M) \in (\mathbb{Q}(\pi) \otimes_{\mathbb{Q}} \mathbb{C})^{\times}$: Deligne's periods attached to M.
- Motivic L-function of M:

$$L(M,s) = \left(L^{(\infty)}\left(s - \frac{w}{2}, \pi^{\sigma}\right)\right)_{\sigma},$$

where σ runs over a complete set of coset representatives of $\mathrm{Aut}(\mathbb{C})/\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\pi))$.

Conjecture (Deligne (1977))

Let $m \in \mathbb{Z}$ be a critical point for M. For any finite order character χ of $\mathbb{Q}^{\times}\backslash\mathbb{A}^{\times}$, we have

$$\frac{L(M \otimes \chi, m)}{(2\pi\sqrt{-1})^{2m} \cdot G(\chi)^2 \cdot c^{(-1)^m \mathrm{sgn}(\chi)}(M)} \in \mathbb{Q}(\pi).$$

Here $G(\chi)$ is the Gauss sum of χ.

We have the following automorphic analogue of Deligne's conjecture.

Theorem (Grobner–Raghuram (2014), Januszewski (2016), Jiang–Sun–Tian (2019))

There exist $c^{\pm}(\pi) \in \mathbb{C}^{\times}$ *such that*

$$\left(\frac{L(\frac{1}{2}+m, \pi \times \chi)}{G(\chi)^2 \cdot c^{(-1)^m \mathrm{sgn}(\chi)}(\pi)} \right)^{\sigma} = \frac{L(\frac{1}{2}+m, \pi^{\sigma} \times \chi^{\sigma})}{G(\chi^{\sigma})^2 \cdot c^{(-1)^m \mathrm{sgn}(\chi)}(\pi^{\sigma})}$$

for all $\sigma \in \mathrm{Aut}(\mathbb{C})$, *all critical points* $\frac{1}{2}+m \in \frac{1}{2}+\mathbb{Z}$ *of* $L(s,\pi)$, *and any finite order character* χ *of* $\mathbb{Q}^{\times}\backslash\mathbb{A}^{\times}$.

Remark

The theorem is a special case of the results on the algebraicity of the critical values of the twisted standard L-functions of irr. regular algebraic, symplectic cusp. auto. rep. of GL_{2n}.

Yoshida's period relation

- π^{hol}: the unique irr. holo. cusp. auto. rep. of $\mathrm{GSp}_4(\mathbb{A})$ such that

$$\pi_f^{\mathrm{hol}} \simeq \pi_f.$$

- f_{hol}: a non-zero vector-valued holomorphic cusp form associated to π^{hol} normalized so that its Fourier coefficients belong to $\mathbb{Q}(\pi)$.
- $c^{\pm}(\mathrm{Sym}^2(M))$: Deligne's periods attached to $\mathrm{Sym}^2(M)$.

Theorem (Yoshida (2001))

We have

$$\frac{c^{+}(\mathrm{Sym}^2(M))}{(2\pi\sqrt{-1})^{6-3\lambda_1+3\lambda_2} \cdot c^{+}(M) \cdot c^{-}(M) \cdot (\|f_{\mathrm{hol}}^{\sigma}\|)_{\sigma}} \in \mathbb{Q}(\pi).$$

- $L(s, \pi, \mathrm{Ad})$: the adjoint L-function of π.
- Motivic L-function of $\mathrm{Sym}^2(M)$:

$$L(\mathrm{Sym}^2(M), s) = \left(L^{(\infty)}(s - w, \pi^\sigma, \mathrm{Ad})\right)_\sigma .$$

Conjecture (Deligne (1977))

Let $m \in \mathbb{Z}$ be a critical point for M. We have

$$\frac{L(\mathrm{Sym}^2(M), m)}{(2\pi\sqrt{-1})^{d^{(-1)^m} m} \cdot c^{(-1)^m}(\mathrm{Sym}^2(M))} \in \mathbb{Q}(\pi).$$

Here $d^+ = 6$ and $d^- = 4$.

In particular, when m is evem, we have

$$\frac{L(\mathrm{Sym}^2(M), m)}{(2\pi\sqrt{-1})^{6+6m-3\lambda_1+3\lambda_2} \cdot c^+(M) \cdot c^-(M) \cdot (\|f^\sigma_{\mathrm{hol}}\|)_\sigma} \in \mathbb{Q}(\pi)$$

Main result

Following is our main result, which can be regarded as an automorphic analogue of Yoshida's period relation.

Theorem (C.-)

Assume that $\lambda_1 + \lambda_2 \geq 4$ *and* $\lambda_2 \leq -5$. *For* $\sigma \in \mathrm{Aut}(\mathbb{C})$, *we have*

$$\left(\frac{L(1, \pi, \mathrm{Ad})}{\pi^3 \cdot c^+(\pi) \cdot c^-(\pi) \cdot \|f_{\mathrm{hol}}\|} \right)^{\sigma} = \frac{L(1, \pi^\sigma, \mathrm{Ad})}{\pi^3 \cdot c^+(\pi^\sigma) \cdot c^-(\pi^\sigma) \cdot \|f^{\sigma}_{\mathrm{hol}}\|}.$$

Remark

(1) The result holds for any irr. (tempered) stable cusp. auto. rep. π of $\mathrm{GSp}_4(\mathbb{A})$ with trivial central character such that π_∞ is a discrete series representation.
(e.g. $\pi = \pi_F$ with Hecke eigenform $F \in S_{2-\lambda_2, \lambda_1+\lambda_2-2}(\mathrm{Sp}_4(\mathbb{Z}))$.)

(2) In case $\lambda_1 + \lambda_2 = 2$, π_p is unramified for all primes p, and there exists a quadratic character χ of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$ with $\chi_\infty(-1) = -1$ such that $L\left(\frac{1}{2}, \pi\right) L\left(\frac{1}{2}, \pi \times \chi\right) \neq 0$. Then the theorem follows from the explicit refinement of Böcherer's conjecture proved by Furusawa–Morimoto.

Symmetric sixth power L-functions for GL_2

- $\tau \subset \mathcal{A}_0(\mathrm{PGL}_2(\mathbb{A}))$: non-dihedral with $\tau_\infty \simeq D(\kappa)$ for some $\kappa \geq 2$.
- $\pi \subset \mathcal{A}_0(\mathrm{GSp}_4(\mathbb{A}))$: the automorphic descent of $\mathrm{Sym}^3\tau$.

Corollary

Assume $\kappa \geq 6$. For $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have

$$\left(\frac{L(1,\tau,\mathrm{Sym}^6)}{\pi^6 \cdot \|f_\tau\|^3 \cdot \|F_\tau\|}\right)^\sigma = \frac{L(1,\tau^\sigma,\mathrm{Sym}^6)}{\pi^6 \cdot \|f_\tau^\sigma\|^3 \cdot \|F_\tau^\sigma\|}.$$

Here

- *f_τ is the normalized newform of τ,*
- *F_τ is a non-zero vector-valued holomorphic cusp form associated to π^{hol} normalized so that its Fourier coefficients belong to $\mathbb{Q}(\tau)$.*

Theorem (Morimoto)

For $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have

$$\left(\frac{\|F_\tau\|}{\|f_\tau\|^3}\right)^\sigma = \pm\frac{\|F_\tau^\sigma\|}{\|f_\tau^\sigma\|^3}.$$

Combining the corollary with Morimoto's result, we obtain the following theorem.

Theorem

For $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have

$$\left(\frac{L(1,\tau,\mathrm{Sym}^6)}{\pi^6 \cdot \|f_\tau\|^6}\right)^\sigma = \pm\frac{L(1,\tau^\sigma,\mathrm{Sym}^6)}{\pi^6 \cdot \|f_\tau^\sigma\|^6}.$$

Remark

(1) Morimoto proved in a different way that
- the ratio is in $\overline{\mathbb{Q}}$,
- elliptic modular form $\to$ Hilbert modular form,
- all critical values,
- twisted symmetric fourth power L-function.

(2) When τ has level 1 and F_τ is a Hecke eigenform, we call F_τ the Kim−Ramakrishnan−Shahidi lift of f_τ. Katsurada−Takemori conjectured that, after suitably normalized, a prime ideal dividing the ratio but not dividing $\Gamma(2\kappa)$ gives a congruence between F_τ and non K-R-S lift.

Proof of the main result

We show that there exist $\Omega_\pm^W(\pi) \in \mathbb{C}^\times$, call the Whittaker periods of π, satisfying the following assertions:

(1) For $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have

$$\left(\frac{L\left(\frac{1}{2}+m,\pi\right) L\left(\frac{\lambda_1+\lambda_2-1}{2},\pi\times\chi\right)}{\pi^{-2\lambda_1}\cdot G(\chi)^2\cdot\Omega_+^W(\pi)} \right)^\sigma = \frac{L\left(\frac{1}{2}+m,\pi^\sigma\right) L\left(\frac{\lambda_1+\lambda_2-1}{2},\pi^\sigma\times\chi^\sigma\right)}{\pi^{-2\lambda_1}\cdot G(\chi^\sigma)^2\cdot\Omega_+^W(\pi^\sigma)}$$

for any finite order character χ of $\mathbb{Q}^\times\backslash\mathbb{A}^\times$ and critical points $\frac{1}{2}+m$ such that $(-1)^{m+\frac{\lambda_1+\lambda_2}{2}}\chi_\infty(-1) = 1$.

(2) For $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have

$$\left(\frac{L(m,\pi\times\tau)}{\pi^{-\lambda_1+\lambda_2}\cdot G(\chi_\tau)^2\cdot\Omega_-^W(\pi)\cdot\|f_\tau\|} \right)^\sigma = \frac{L(m,\pi^\sigma\times\tau^\sigma)}{\pi^{-\lambda_1+\lambda_2}\cdot G(\chi_\tau^\sigma)^2\cdot\Omega_-^W(\pi^\sigma)\cdot\|f_{\tau^\sigma}\|}$$

for any irr. cusp. auto. rep. τ of $\mathrm{GL}_2(\mathbb{A})$ satisfying:

(i) $\omega_\tau = \chi_\tau|\ |_{\mathbb{A}}^r$ for some finite order character χ_τ of $\mathbb{Q}^\times\backslash\mathbb{A}^\times$ and $r \in \mathbb{Z}$,

(ii) $\tau_\infty \otimes |\ |_{\mathbb{R}}^{-r/2} \simeq D(\ell)$ for some $\lambda_1+\lambda_2+1 \le \ell \le \lambda_1$ with $\ell \equiv r \pmod 2$,

and any critical points $m \in \mathbb{Z}$ with $m > -\frac{r}{2}$.

(3) For $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have

$$\left(\frac{\|f_\pi\|}{\Omega_+^W(\pi)\cdot\Omega_-^W(\pi)}\right)^\sigma = \frac{\|f_{\pi^\sigma}\|}{\Omega_+^W(\pi^\sigma)\cdot\Omega_-^W(\pi^\sigma)}.$$

Here f_π is the normalized newform of π (to be explained later).

(1) + results of Januszewski, Jiang–Sun–Tian: if $\lambda_1+\lambda_2 \geq 4$, then

$$\left(\frac{\Omega_+^W(\pi)}{\pi^{2\lambda_1}\cdot c^+(\pi)\cdot c^-(\pi)}\right)^\sigma = \frac{\Omega_+^W(\pi^\sigma)}{\pi^{2\lambda_1}\cdot c^+(\pi^\sigma)\cdot c^-(\pi^\sigma)}.$$

(2) + results of Furusawa, Böcherer–Heim, Pitale–Schmidt, Saha, Morimoto: if $\lambda_2 \leq -5$, then

$$\left(\frac{\Omega_-^W(\pi)}{\pi^{4+\lambda_1-\lambda_2}\cdot\|f_{\mathrm{hol}}\|}\right)^\sigma = \frac{\Omega_-^W(\pi^\sigma)}{\pi^{4+\lambda_1-\lambda_2}\cdot\|f_{\mathrm{hol}}^\sigma\|}.$$

Therefore, by (3) we have

$$\left(\frac{\|f_\pi\|}{\pi^{4+3\lambda_1-\lambda_2}\cdot c^+(\pi)\cdot c^-(\pi)\cdot\|f_{\mathrm{hol}}\|}\right)^\sigma = \frac{\|f_{\pi^\sigma}\|}{\pi^{4+3\lambda_1-\lambda_2}\cdot c^+(\pi^\sigma)\cdot c^-(\pi^\sigma)\cdot\|f_{\mathrm{hol}}^\sigma\|}.$$

Our main theorem then follows from the following result:

- For $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have

$$\left(\frac{L(1,\pi,\mathrm{Ad})}{\pi^{-1-3\lambda_1+\lambda_2}\cdot \|f_\pi\|}\right)^\sigma = \frac{L(1,\pi^\sigma,\mathrm{Ad})}{\pi^{-1-3\lambda_1+\lambda_2}\cdot \|f_{\pi^\sigma}\|}.$$

  C.−Ichino (2019): We computed the explicit value for the ratio when π has square-free paramodular conductor.

In the rest of this talk, we

- sketch the construction of the Whittaker periods $\Omega_\pm^W(\pi)$,
- sketch the proof of the corresponding algebraicity results for critical L-values.

Notation

- For $(k_1, k_2) \in \mathbb{Z}^2$ with $k_1 \geq k_2$, let $(\rho_{(k_1,k_2)}, V_{(k_1,k_2)})$ be the irr. alg. rep. of $\mathrm{U}(2)$ defined by

$$\rho_{(k_1,k_2)} = \mathrm{Sym}^{k_1-k_2} \otimes \det{}^{k_2},$$
$$V_{(k_1,k_2)} = \langle X^i Y^{k_1-k_2-i} \mid 0 \leq i \leq k_1 - k_2 \rangle_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}.$$

- Maximal unipotent subgroup of GSp_4:

$$U = \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \end{pmatrix} \in \mathrm{GSp}_4 \right\}.$$

- $\psi_U : U(\mathbb{Q}) \backslash U(\mathbb{A}) \longrightarrow \mathbb{C}^\times$ defined by

$$\psi_U \left(\begin{pmatrix} 1 & x & * & * \\ 0 & 1 & * & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x & 1 \end{pmatrix} \right) = \psi(-x - y),$$

where $\psi : \mathbb{Q} \backslash \mathbb{A} \longrightarrow \mathbb{C}^\times$ so that $\psi_\infty(x) = e^{2\pi\sqrt{-1}\,x}$.

- For $\varphi \in \pi$, define the global Whittaker function

$$W_{\varphi,\psi_U}(g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug)\overline{\psi_U(u)}\, du.$$

- $\mathcal{W}(\pi_v, \psi_{U,v})$: the space of Whittaker functions of π_v with respect to $\psi_{U,v}$.
- $\mathcal{W}(\pi_f, \psi_{U,f}) = \bigotimes_p' \mathcal{W}(\pi_p, \psi_{U,p})$.
- For $\sigma \in \mathrm{Aut}(\mathbb{C})$, define the σ-linear isomorphism of $\mathrm{GSp}_4(\mathbb{A}_f)$-modules:

$$t_\sigma : \mathcal{W}(\pi_f, \psi_{U,f}) \longrightarrow \mathcal{W}(\pi_f^\sigma, \psi_{U,f}),$$
$$t_\sigma W(g) = \sigma\left(W\left(\mathrm{diag}(u^{-2}, u^{-1}, u, 1)g\right)\right).$$

Here $\sigma|_{\mathbb{Q}^{\mathrm{ab}}} = \mathrm{rec}(a \cdot u)$ with $a \cdot u \in \mathbb{R}_{>0}^\times \cdot \widehat{\mathbb{Z}}^\times$ and $\mathrm{rec} : \mathbb{Q}^\times \backslash \mathbb{A}^\times \longrightarrow \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ is the geometrically normalized reciprocity map.

- Recall

$$\pi_\infty|_{\mathrm{Sp}_4(\mathbb{R})} = D_{(\lambda_1,\lambda_2)} \oplus D_{(-\lambda_2,-\lambda_1)},$$

for some $(\lambda_1,\lambda_2) \in \mathbb{Z}^2$ such that $1-\lambda_1 \leq \lambda_2 \leq -1$.

- Write

$$\pi^+ = (\pi \otimes_{\mathbb{C}} V_{(\lambda_1,\lambda_2)})^{\mathrm{U}(2)}, \quad \pi^- = (\pi \otimes_{\mathbb{C}} V_{(-\lambda_2,-\lambda_1)})^{\mathrm{U}(2)}.$$

Note that $\pi^+ \simeq \pi^- \simeq \pi_f$ as $\mathrm{GSp}_4(\mathbb{A}_f)$-modules.

- For $f \in \pi^+$, $h \in \pi^-$, we have

$$f = \sum_{i=0}^{\lambda_1-\lambda_2} (-1)^i \binom{\lambda_1-\lambda_2}{i} \cdot P_i^+(f) \otimes X^{\lambda_1-\lambda_2-i}Y^i,$$

$$h = \sum_{i=0}^{\lambda_1-\lambda_2} \binom{\lambda_1-\lambda_2}{i} \cdot P_i^-(h) \otimes X^iY^{\lambda_1-\lambda_2-i}$$

for some uniquely determined $P_i^+(f), P_i^-(h) \in \pi$ for $0 \leq i \leq \lambda_1-\lambda_2$.

Let $0 \leq i \leq \lambda_1 - \lambda_2$.

- $W_i \in \mathcal{W}(\pi_\infty, \psi_{U,\infty})$: in the minimal $\mathrm{U}(2)$-type of $D_{(-\lambda_2,-\lambda_1)}$ with weight $(-\lambda_1 + i, -\lambda_2 - i)$ normalized so that (following T. Moriyama)

$$
\begin{aligned}
&W_i(1) \\
&= (2\sqrt{-1})^{\frac{3\lambda_1+\lambda_2}{2} - i} \pi^{-\frac{1}{2}} e^{-2\pi} \\
&\times \int_{c_1-\sqrt{-1}\infty}^{c_1+\sqrt{-1}\infty} \frac{ds_1}{2\pi\sqrt{-1}} \int_{c_2-\sqrt{-1}\infty}^{c_2+\sqrt{-1}\infty} \frac{ds_2}{2\pi\sqrt{-1}} (4\pi^3)^{\frac{-s_1+\lambda_1+1-i}{2}} (4\pi)^{\frac{-s_2+\lambda_2+i}{2}} \\
&\quad \times \Gamma\left(\frac{s_1+s_2-2\lambda_2+1}{2}\right) \Gamma\left(\frac{s_1+s_2+1}{2}\right) \Gamma\left(\frac{s_1}{2}\right) \Gamma\left(\frac{-s_2}{2}\right) \frac{\Gamma(s_1+i)}{\Gamma(s_1)},
\end{aligned}
$$

  where $c_1, c_2 \in \mathbb{R}$ satisfy $c_1 + c_2 + 1 > 0$ and $c_1 > 0 > c_2$.

- $\overline{W}_i \in \mathcal{W}(\pi_\infty, \overline{\psi}_{U,\infty})$: in the minimal $\mathrm{U}(2)$-type of $D_{(\lambda_1,\lambda_2)}$ with weight $(\lambda_1 - i, \lambda_2 + i)$.

- Define $\mathrm{GSp}_4(\mathbb{A}_f)$-module isomorphisms

$$\pi^+ \longrightarrow \mathcal{W}(\pi_f, \psi_{U,f}), \quad f \longmapsto W_f^+$$
$$\pi^- \longrightarrow \mathcal{W}(\pi_f, \overline{\psi}_{U,f}), \quad h \longmapsto W_h^-$$

  by

$$W_{P_i^+(f),\psi_U} = W_i \cdot W_f^+, \quad W_{P_i^-(h),\overline{\psi}_U} = \overline{W}_i \cdot W_h^-$$

  for $0 \leq i \leq \lambda_1 - \lambda_2$.

- Define the σ-linear isomorphisms of $\mathrm{GSp}_4(\mathbb{A}_f)$-modules

$$\pi^+ \longrightarrow (\pi^\sigma)^+, \quad f \longmapsto f^\sigma$$
$$\pi^- \longrightarrow (\pi^\sigma)^-, \quad h \longmapsto h^\sigma$$

  by

$$W_{f^\sigma}^+ = t_\sigma W_f^+, \quad W_{h^\sigma}^- = t_\sigma W_h^-.$$

- $f_\pi \in \pi^+$: the normalized newform of π defined so that $W_{f_\pi}^+ \in \mathcal{W}(\pi_f, \psi_{U,f})$ is the paramodular newform with $W_{f_\pi}^+(1) = 1$. It is clear that $f_\pi^\sigma = f_{\pi^\sigma}$ for $\sigma \in \mathrm{Aut}(\mathbb{C})$.

Rational structures via the cohomology

- Put $K_\infty = \mathbb{R}^\times \cdot \mathrm{U}(2) \subset \mathrm{GSp}_4(\mathbb{R})$ and (cf. Oshima's and Horinaga's talk)

$$\mathfrak{p}^- = \left\{ \begin{pmatrix} \sqrt{-1}A & A \\ A & \sqrt{-1}A \end{pmatrix} \,\middle|\, A = {}^t\!A \in \mathrm{M}_2(\mathbb{Q}) \right\} \otimes_{\mathbb{Q}} \mathbb{C} \subset \mathrm{Lie}(\mathrm{GSp}_4(\mathbb{R}))_{\mathbb{C}}.$$

- Let (ρ, V_ρ) be an irr. alg. rep. of K_∞. Consider the complexes with respect to the Lie algebra differential operator:

$$C^q_{\mathrm{sia},\rho} = \left(C^\infty_{\mathrm{sia}}(\mathrm{GSp}_4(\mathbb{Q})\backslash \mathrm{GSp}_4(\mathbb{A})) \otimes_{\mathbb{C}} \bigwedge^q (\mathfrak{p}^-)^* \otimes_{\mathbb{C}} V_\rho \right)^{K_\infty},$$

$$C^q_{\mathrm{rda},\rho} = \left(C^\infty_{\mathrm{rda}}(\mathrm{GSp}_4(\mathbb{Q})\backslash \mathrm{GSp}_4(\mathbb{A})) \otimes_{\mathbb{C}} \bigwedge^q (\mathfrak{p}^-)^* \otimes_{\mathbb{C}} V_\rho \right)^{K_\infty}$$

  for $q \geq 0$.

- $H^q(\mathcal{V}^{\mathrm{can}}_\rho)$ and $H^q(\mathcal{V}^{\mathrm{sub}}_\rho)$: the q-th cohomology groups with respect to the complexes $C^*_{\mathrm{sia},\rho}$ and $C^*_{\mathrm{rda},\rho}$, respectively.
- $H^q_!(\mathcal{V}_\rho)$: the image of the morphism $H^q(\mathcal{V}^{\mathrm{sub}}_\rho) \longrightarrow H^q(\mathcal{V}^{\mathrm{can}}_\rho)$ induced by the inclusion $C^*_{\mathrm{rda},\rho} \longrightarrow C^*_{\mathrm{sia},\rho}$.

Rational structures via the cohomology

Theorem (Harris, Milne)

(1) *$H^q(\mathcal{V}_\rho^{\mathrm{can}})$ and $H^q(\mathcal{V}_\rho^{\mathrm{sub}})$ are admissible $\mathrm{GSp}_4(\mathbb{A}_f)$-modules and have canonical rational structures over $\mathbb{Q}$.*

(2) *$H_!^q(\mathcal{V}_\rho)$ is semisimple.*

(3) *For $\sigma \in \mathrm{Aut}(\mathbb{C})$, conjugation by σ induces natural σ-linear $\mathrm{GSp}_4(\mathbb{A}_f)$-module isomorphism:*

$$T_\sigma : H^q(\mathcal{V}_\rho^{\mathrm{can}}) \longrightarrow H^q(\mathcal{V}_\rho^{\mathrm{can}}).$$

Similar assertion holds for $H^q(\mathcal{V}_\rho^{\mathrm{sub}})$ and $H_!^q(\mathcal{V}_\rho)$.

(4) *We have a natural injective homomorphism of $\mathrm{GSp}_4(\mathbb{A}_f)$-modules*

$$\mathrm{cl} : \left(\mathcal{A}_0(\mathrm{GSp}_4(\mathbb{A}), \rho) \otimes_{\mathbb{C}} \bigwedge^q (\mathfrak{p}^-)^* \otimes_{\mathbb{C}} V_\rho \right)^{K_\infty} \longrightarrow H_!^q(\mathcal{V}_\rho)$$

for each $q \in \mathbb{Z}_{\geq 0}$. Here $\mathcal{A}_0(\mathrm{GSp}_4(\mathbb{A}), \rho)$ is the space of cusp forms on $\mathrm{GSp}_4(\mathbb{A})$ which are eigenfunctions of the Casimir operator of $\mathrm{GSp}_4(\mathbb{R})$ with certain eigenvalue depending on ρ.

Rational structures via the cohomology

We apply the results of Harris to (ρ, V_ρ) equal to

$$(\rho^+, V^+) := (\rho_{(\lambda_1-3,\lambda_2-1)}, V_{(\lambda_1-3,\lambda_2-1)}),$$
$$(\rho^-, V^-) := (\rho_{(-\lambda_2-2,-\lambda_1)}, V_{(-\lambda_2-2,-\lambda_1)}).$$

In these two cases, π occurs in $\mathcal{A}_0(\mathrm{GSp}_4(\mathbb{A}), \rho)$.

Proposition

The homorphisms cl *induce isomorphisms of* $\mathrm{GSp}_4(\mathbb{A}_f)$*-modules:*

$$\mathrm{cl}: \left[\pi \otimes_{\mathbb{C}} \bigwedge^2 (\mathfrak{p}^-)^* \otimes_{\mathbb{C}} V^+\right]^{K_\infty} \longrightarrow H^2_!(\mathcal{V}_{\rho^+})[\pi_f] \simeq \pi_f,$$

$$\mathrm{cl}: \left[\pi \otimes_{\mathbb{C}} \bigwedge^1 (\mathfrak{p}^-)^* \otimes_{\mathbb{C}} V^-\right]^{K_\infty} \longrightarrow H^1_!(\mathcal{V}_{\rho^-})[\pi_f] \simeq \pi_f.$$

Remark

We use Arthur's multiplicity formula to deduce that

$$\mathcal{A}_0(\mathrm{GSp}_4(\mathbb{A}))[\pi_f] = \pi \oplus \pi^{\mathrm{hol}}.$$

Fix $\mathbb{Q}$-rational injective K_∞-equivariant homomorphisms:

$$V_{(\lambda_1,\lambda_2)} \longrightarrow \bigwedge^2 (\mathfrak{p}^-)^* \otimes_{\mathbb{C}} V^+ \Longrightarrow \mathrm{cl} : \pi^+ \longrightarrow H_!^2(\mathcal{V}_{\rho^+})[\pi_f],$$

$$V_{(-\lambda_2,-\lambda_1)} \longrightarrow \bigwedge^1 (\mathfrak{p}^-)^* \otimes_{\mathbb{C}} V^- \Longrightarrow \mathrm{cl} : \pi^- \longrightarrow H_!^1(\mathcal{V}_{\rho^-})[\pi_f].$$

Note that the $\mathbb{Q}$-rational embeddings are unique up to homotheties over $\mathbb{Q}^\times$.

Lemma

There exist $\Omega_+^W(\pi)$, $\Omega_-^W(\pi) \in \mathbb{C}^\times$, unique up to $\mathbb{Q}(\pi)^\times$, such that

$$\frac{\mathrm{cl}\left(\pi^{+,\,\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\pi))}\right)}{\Omega_+^W(\pi)} = H_!^2(\mathcal{V}_{\rho^+})[\pi_f]^{\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\pi))},$$

$$\frac{\mathrm{cl}\left(\pi^{-,\,\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\pi))}\right)}{\Omega_-^W(\pi)} = H_!^1(\mathcal{V}_{\rho^-})[\pi_f]^{\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\pi))}.$$

We call $\Omega_\pm^W(\pi)$ the Whittaker periods of π. By the uniqueness up to $\mathbb{Q}(\pi)^\times$, we can normalize the Whittaker periods so that

$$T_\sigma\left(\frac{\mathrm{cl}(f)}{\Omega_\varepsilon^W(\pi)}\right) = \frac{\mathrm{cl}(f^\sigma)}{\Omega_\varepsilon^W(\pi^\sigma)}, \quad f \in \pi^\varepsilon.$$

Theorem (C.-, in progress)

(1) *For $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have*

$$\left(\frac{L\left(\frac{1}{2}+m,\pi\right) L\left(\frac{\lambda_1+\lambda_2-1}{2},\pi\times\chi\right)}{\pi^{-2\lambda_1}\cdot G(\chi)^2\cdot \Omega_+^W(\pi)} \right)^\sigma = \frac{L\left(\frac{1}{2}+m,\pi^\sigma\right) L\left(\frac{\lambda_1+\lambda_2-1}{2},\pi^\sigma\times\chi^\sigma\right)}{\pi^{-2\lambda_1}\cdot G(\chi^\sigma)^2\cdot \Omega_+^W(\pi^\sigma)}$$

for any finite order character χ of $\mathbb{Q}^\times\backslash\mathbb{A}^\times$ and critical points $\frac{1}{2}+m$ such that $(-1)^{m+\frac{\lambda_1+\lambda_2}{2}}\chi_\infty(-1)=1$.

(2) *For $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have*

$$\left(\frac{L(m,\pi\times\tau)}{\pi^{-\lambda_1+\lambda_2}\cdot G(\chi_\tau)^2\cdot \Omega_-^W(\pi)\cdot \|f_\tau\|} \right)^\sigma = \frac{L(m,\pi^\sigma\times\tau^\sigma)}{\pi^{-\lambda_1+\lambda_2}\cdot G(\chi_\tau^\sigma)^2\cdot \Omega_-^W(\pi^\sigma)\cdot \|f_{\tau^\sigma}\|}$$

for any irr. cusp. auto. rep. τ of $\mathrm{GL}_2(\mathbb{A})$ satisfying:

- (i) $\omega_\tau = \chi_\tau|\ |_{\mathbb{A}}^r$ *for some finite order character χ_τ of $\mathbb{Q}^\times\backslash\mathbb{A}^\times$ and $r\in\mathbb{Z}$,*
- (ii) $\tau_\infty\otimes|\ |_{\mathbb{R}}^{-r/2}\simeq D(\ell)$ *for some $\lambda_1+\lambda_2+1\leq\ell\leq\lambda_1$ with $\ell\equiv r\ (\mathrm{mod}\,2)$,*

and any critical points $m\in\mathbb{Z}$ with $m>-\frac{r}{2}$.

We sketch the proof of (2). Write $\omega = \omega_\tau$ and $\chi = \chi_\tau$.

- Let $G' = \{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 \mid \det(g_1) = \det(g_2)\}$ and we regard it as a subgroup of GSp_4 by the embedding

$$G' \longrightarrow \mathrm{GSp}_4, \quad \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \longmapsto \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix}.$$

Ingredients:

(1) Integral representation of $L(s, \pi \times \tau)$.

(2) Cohomological interpretation of the global integral.

(3) Local zeta integrals:

- Explicit calculation of the archimedean local zeta integral.
- Galois equivariance property of the p-adic local zeta integral.

(1) Integral representation of $L(s, \pi \times \tau)$ (Piatetski-Shapiro–Soudry):

- For $\varphi_1 \in \pi$, $\varphi_2 \in \tau$, and $f_s \in \mathrm{Ind}_{B(\mathbb{A})}^{\mathrm{GL}_2(\mathbb{A})}(|\ |_{\mathbb{A}}^{s-\frac{1}{2}} \boxtimes |\ |_{\mathbb{A}}^{-s+\frac{1}{2}}\omega^{-1})$ be a good section, we have the basic identity

$$Z(\varphi_1, \varphi_2, E(f_s)) := \int_{\mathbb{A}^\times G'(\mathbb{Q})\backslash G'(\mathbb{A})} \varphi_1(g) E(g_1; f_s) \varphi_2(g_2)\, dg$$
$$= \int_{\mathbb{A}^\times N'(\mathbb{A})\backslash G'(\mathbb{A})} W_{\varphi_1, \psi_U}(g) f_s(g_1) W_{\varphi_2, \psi}(g_2)\, dg.$$

  Here dg is the Tamagawa measure on $\mathbb{A}^\times\backslash G'(\mathbb{A})$.

- For $\kappa \geq 1$ and $\Phi \in \mathcal{S}(\mathbb{A}_f^2)$, let $E_s^{[\kappa]}(\omega, \Phi) = E(f_{\omega, \Phi^{[\kappa]}\otimes\Phi, s})$, where

$$f_{\omega, \Phi^{[\kappa]}\otimes\Phi, s}(g) = |\det(g)|_{\mathbb{A}}^{s} \int_{\mathbb{A}^\times} (\Phi^{[\kappa]} \otimes \Phi)((0, t)g)\omega(t)|t|_{\mathbb{A}}^{2s}\, d^\times t$$

  and

$$\Phi^{[\kappa]}(x, y) = 2^{-\kappa}(x + \sqrt{-1}y)^\kappa e^{-\pi(x^2+y^2)}.$$

  Then $E_s^{[\kappa]}(\omega, \Phi)|_{s=\frac{\kappa - r}{2}}$ is a holo. auto. form of weight κ.

(2) Cohomological interpretation of the global integral.

- Let $m \in \mathbb{Z}$ with $-\frac{r}{2} < m \leq \frac{\ell-\lambda_1-\lambda_2-r}{2}$ (right-half critical points).

$$H^1_!(\mathcal{V}_{\rho^-}) \otimes_{\mathbb{C}} H^1_!(\mathcal{V}_{(\ell-2;-r)}) \otimes_{\mathbb{C}} H^0(\mathcal{V}^{\mathrm{can}}_{(\lambda_1+\lambda_2-\ell;r)}) \longrightarrow \mathbb{C},$$

$$[f_1] \otimes [f_2] \otimes [f_3] \longmapsto Z\left(P^-_{\ell-\lambda_2}(f_1),\ f_2,\ X_+^{\frac{\ell-\lambda_1-\lambda_2-r}{2}-m} \cdot f_3\right)$$

is a well-defined $\mathbb{Q}$-rational trilinear form. Here

- $\mathcal{V}_{(\kappa;r)}$: the automorphic line bundle on $\mathcal{M}_{\mathrm{GL}_2}$ associated to the character

$$a \cdot e^{\sqrt{-1}\theta} \longmapsto a^r \cdot e^{\kappa\sqrt{-1}\theta}$$

of $\mathbb{R}^\times \cdot U(1)$.

- $X_+ = -\frac{1}{4}\begin{pmatrix} \sqrt{-1} & -1 \\ -1 & -\sqrt{-1} \end{pmatrix} \in \mathrm{Lie}(\mathrm{GL}_2(\mathbb{R}))_{\mathbb{C}}$ is the weight raising differential operator.

- For $\varphi_1 \in \pi^-$, $\varphi_2 \in \tau$ with weight $-\ell$, and $\Phi \in \mathcal{S}(\mathbb{A}_f^2)$, we have

$$T_\sigma\left(\frac{[\varphi_1]}{\Omega_-^W(\pi)}\right) = \frac{[\varphi_1^\sigma]}{\Omega_-^W(\pi^\sigma)} \in H_!^1(\mathcal{V}_{\rho^-})[\pi_f^\sigma],$$

$$T_\sigma\left(\frac{[\varphi_2]}{\|f_\tau\|(2\pi\sqrt{-1})^{\frac{\ell-r}{2}}}\right) = \frac{[\varphi_2^\sigma]}{\|f_{\tau^\sigma}\|(2\pi\sqrt{-1})^{\frac{\ell-r}{2}}} \in H_!^1(\mathcal{V}_{(\ell-2;-r)})[\tau_f^\sigma],$$

$$T_\sigma\left(\frac{[E_s^{[2m+r]}(\omega,\Phi)|_{s=m}]}{G(\chi)(2\pi\sqrt{-1})^{-m}}\right) = \frac{[E_s^{[2m+r]}(\omega^\sigma,\Phi^\sigma)|_{s=m}]}{G(\chi^\sigma)(2\pi\sqrt{-1})^{-m}} \in H^0(\mathcal{V}^{\mathrm{can}}_{(\lambda_1+\lambda_2-\ell;r)}).$$

We conclude that

$$\left(\frac{Z\left(P^-_{\ell-\lambda_2}(\varphi_1),\, \varphi_2,\, X_+^{\frac{\ell-\lambda_1-\lambda_2-r}{2}-m}\cdot E_s^{[2m+r]}(\omega,\Phi)|_{s=m}\right)}{\Omega_-^W(\pi)\cdot\|f_\tau\|(2\pi\sqrt{-1})^{\frac{\ell-r}{2}}\cdot G(\chi)(2\pi\sqrt{-1})^{-m}}\right)^\sigma$$

$$= \frac{Z\left(P^-_{\ell-\lambda_2}(\varphi_1^\sigma),\, \varphi_2^\sigma,\, X_+^{\frac{\ell-\lambda_1-\lambda_2-r}{2}-m}\cdot E_s^{[2m+r]}(\omega^\sigma,\Phi^\sigma)|_{s=m}\right)}{\Omega_-^W(\pi^\sigma)\cdot\|f_{\tau^\sigma}\|(2\pi\sqrt{-1})^{\frac{\ell-r}{2}}\cdot G(\chi^\sigma)(2\pi\sqrt{-1})^{-m}}$$

(3) Local zeta integrals:

- Let
  (i) $\overline{W}_{\ell-\lambda_2} \in \mathcal{W}(\pi_\infty, \overline{\psi}_{U,\infty})$: normalized with weight $(\lambda_1+\lambda_2-\ell, \ell)$,
  (ii) $W'_{-\ell} \in \mathcal{W}(\tau_\infty, \overline{\psi}_\infty)$: weight $-\ell$ with $W'_{-\ell}(1) = e^{-2\pi}$,
  (iii) $f_{\omega_\infty, \Phi^{[\ell-\lambda_1-\lambda_2]}, s}$: normalized with weight $\ell-\lambda_1-\lambda_2$.

  We have

$$Z_\infty(\overline{W}_{\ell-\lambda_2},\ W'_{-\ell},\ f_{\omega_\infty, \Phi^{[\ell-\lambda_1-\lambda_2]}, s}) \in (\sqrt{-1})^{\frac{\lambda_1+\lambda_2}{2}} \cdot \pi^{\frac{3\lambda_1-\lambda_2}{2}} \cdot L(s, \pi_\infty \times \tau_\infty) \cdot \mathbb{Q}^\times.$$

- Let p be a prime. Let $W_1 \in \mathcal{W}(\pi_p, \overline{\psi}_{U,p})$, $W_2 \in \mathcal{W}(\tau_p, \overline{\psi}_p)$, and $\Phi \in \mathcal{S}(\mathbb{Q}_p^2)$. For $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have

$$\left(\frac{Z_p(W_1, W_2, f_{\omega_p, \Phi, s})}{\varepsilon(0, \chi_p, \psi_p)}\right)^\sigma = \frac{Z_p(t_\sigma W_1, t_\sigma W_2, f_{\omega_p^\sigma, \Phi^\sigma, s})}{\varepsilon(0, \chi_p^\sigma, \psi_p)},$$

$$L(s, \pi_p \times \tau_p)^\sigma = L(s, \pi_p^\sigma \times \tau_p^\sigma).$$

The assertion then follows from (1)-(3) with good choice of datum

$$(W_1, W_2, \Phi) \in \mathcal{W}(\pi_f, \overline{\psi}_{U,f}) \times \mathcal{W}(\tau_f, \overline{\psi}_f) \times \mathcal{S}(\mathbb{A}_f^2).$$

The end.
Thank you for your attention.