Arthur packets for real symplectic groups and unitary highest weight modules

David Renard, joint work with C. Moeglin

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Introduction

It has been explained in other talks that the following problems are interesting :

Let $G = \mathbf{Sp}(2n, \mathbb{R})$, the real symplectic group, and π an irreducible highest weight unitary representation of G.

(1) Find all Arthur parameters ψ such that π is in the corresponding Arthur packet $\Pi(G, \psi)$.

(2) Show multiplicity one of π in $\Pi(G, \psi)$.

(3) Determine the invariant ρ_π attached to π and ψ (in this case, a character of the component group $A(\psi)$ of the centralizer of ψ in $\widehat{G} = \mathbf{SO}(2n+1)$).
This is the crucial local ingredient in Arthur's multiplicity formula.

Arthur packets, generalities

$\mathbb{G}$: connected reductive algebraic group defined over a local field F, $G = \mathbb{G}(F)$.

Arthur parameter $\psi : W'_F \times \mathbf{SL}_2 \to {}^L G$.
W'_F : Weil-Deligne group of F if F non archimedean, Weil group if F archimedean.
$C(\psi) = \mathrm{Centr}_{\widehat{G}}(\psi)$, $A(\psi) = C(\psi)/C(\psi)_0$ (finite group)

In the late 80's, Arthur suggested that to such ψ there should be attached a "packet" $\Pi(G,\psi)$ of irreducible unitary reps of G, satisfying some properties (endoscopic identities, ...).
To describe these identities, to $\pi \in \Pi(G,\psi)$ is attached ρ_π, a $\mathbb{C}$-valued class function on $A(\psi)$

Rmk. Here we assume $\mathbb{G}$ pure inner form of quasi-split group, otherwise $A(\psi)$ has to be replaced with a covering.

Arthur packets, generalities

Assume that $\mathbb{G}$ is such that the $A(\psi)$ are abelian groups (for classical groups they are 2-groups).

ρ_π is then a $\mathbb{C}$-linear combination of irreducible characters of $A(\psi)$ and to ψ is attached

$$\pi_G^A(\psi)$$

a $\mathbb{C}$-linear combination of irreducible unitary reps of of $G \times A(\psi)$.

In his 2013 book, Arthur established the existence of the $\pi_G^A(\psi)$ when $\mathbb{G}$ is classical (symplectic or special orthogonal). He shows that the endoscopic identities completely characterize these objets (unitary groups, same results by Mok and Kaletha-Minguez-Shin-White)

Arthur packets, generalities

If $\mathbb{G}$ is quasi-split, he shows that the $\mathbb{C}$-linear combination $\pi_G^A(\psi)$ has coefficients in $\mathbb{Z}_{>0}$. So $\pi_G^A(\psi)$ can be seen as a finite length unitary rep. of $G \times A(\psi)$ (Moeglin-R for non quasi-split special orthogonal groups). We can write this in two ways :

$$\pi_G^A(\psi) = \bigoplus_{\eta \in \widehat{A(\psi)}} \pi_G^A(\psi, \eta) \boxtimes \eta$$

where the $\pi_G^A(\psi, \eta)$ are finite length unitary representation of G, or

$$\pi_G^A(\psi) = \bigoplus_{\pi \in \Pi(G,\psi)} \pi \boxtimes \rho_\pi$$

where ρ_π is a finite length unitary representation of $A(\psi)$.

Arthur packets, generalities

Rmks. 1. $\dim \rho_\pi$ is the *multiplicity* of π in the packet $\Pi(G, \psi)$

2. If $s \in A(\psi)$, one can evaluate $\pi_G^A(\psi)$ on s and get a virtual representation of G

$$\pi_G^A(\psi)(s).$$

There are the ones which are transfered from endoscopic groups (endoscopic identities).

3. Endoscopy theory relies on the choice of *transfer factors*. To fix them completely, one needs choice of a *Whittaker datum* for G. Above, the correspondence $\pi \mapsto \rho_\pi$ depends on this choice (in a somewhat non essential way).

Parameters for $G = \mathbf{Sp}(2n, \mathbb{R})$

From now on $F = \mathbb{R}$, $\quad \mathbb{G} = \mathbf{Sp}(2n)$, $\quad {}^L G = \mathbf{SO}(2n+1, \mathbb{C})$.

Arthur parameters :

$$W_{\mathbb{R}} \times \mathbf{SL}_2 \xrightarrow{\psi} \mathbf{SO}(2n+1).$$

Compose this with the standard representation of $\mathbf{SO}(2n+1, \mathbb{C})$

$$\mathbf{SO}(2n+1, \mathbb{C}) \xrightarrow{\mathbf{std}} \mathbf{GL}(2n+1, \mathbb{C}) :$$

Writing again ψ for the composition

$$\psi : W_{\mathbb{R}} \times \mathbf{SL}_2 \longrightarrow \mathbf{SO}(2n+1) \xrightarrow{\mathbf{std}} \mathbf{GL}(2n+1, \mathbb{C})$$

Get a $2n+1$ completely reducible rep of $W_{\mathbb{R}} \times \mathbf{SL}_2$. Let us decompose it.

Irreducible representations of $W_{\mathbb{R}}$

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow W_{\mathbb{R}} \longrightarrow \{\pm 1\} \longrightarrow 1$$

non split extension

- Irreducible representations of $W_{\mathbb{R}}$:

$$\bullet \quad \delta(s_1, s_2), \quad t = s_1 - s_2 \in \mathbb{Z}_{>0}, \quad s = s_1 + s_2 \in i\mathbb{R},$$

2-dimensional, Langlands parameter of relative discrete series of $\mathbf{GL}(2,\mathbb{R})$ with inf. char (s_1, s_2)

$$\bullet \quad \chi(s, \epsilon), \quad \epsilon \in \{0, 1\}, \quad s \in i\mathbb{R},$$

1-dimensional, Langlands parameter of the character $x \mapsto \operatorname{sgn}^\epsilon(x)|x|^s$ of $\mathbf{GL}(1,\mathbb{R})$

Irreducible representations of $W_{\mathbb{R}} \times \mathbf{SL}_2$

$R[a]$: a-dimensional irreducible algebraic rep of $\mathbf{SL}_2$

So irreducible reps of $W_{\mathbb{R}} \times \mathbf{SL}_2$ are of the form :

$$\delta(s_1, s_2) \boxtimes R[a] \quad \text{or} \quad \chi(s, \epsilon) \boxtimes R[a]$$

Parity condition :

$\delta(s_1, s_2) \boxtimes R[a]$ has **good parity** if $s = s_1 + s_2 = 0$, $t = s_1 - s_2$ and a in $\mathbb{Z}_{>0}$ such that $t + a - 1 \cong 0 \mod 2$.

$\chi(s, \epsilon) \boxtimes R[a]$ has **good parity** if $s = 0$, $a - 1 \cong 0 \mod 2$.

Good parity $\iff$ orthogonal rep.

All others irred. rep of $W_{\mathbb{R}} \times \mathbf{SL}_2$ are of **bad parity**.

Decomposition of parameters

Arthur parameter for G

$$\psi : W_{\mathbb{R}} \times \mathbf{SL}_2 \longrightarrow \mathbf{SO}(2n+1) \xrightarrow{\mathbf{std}} \mathbf{GL}(2n+1,\mathbb{C})$$

Suppose that $\tau \boxtimes R[a]$, irreducible rep. of $W_{\mathbb{R}} \times \mathbf{SL}_2$ occurs in the decomposition of ψ, *of bad parity* .

Then $\check{\tau} \boxtimes R[a]$ occurs in ψ, with same multiplicity (if $\tau = \check{\tau}$, the multiplicity is even)

Write the decomposition of ψ into irreducibles as

$$\psi = \psi_{gp} \oplus \psi_{bp}$$

the good parity part and the bad parity part

Decomposition of parameters

where the good parity part decomposes as

$$\psi_{gp} = \psi_u \oplus \psi_d = \bigoplus_{i=1}^{r} \eta_i \boxtimes R[a'_i] \oplus \bigoplus_{j=1}^{s} \delta_{t_j} \boxtimes R[a_j]$$

the *unipotent* part and *discrete* part,

- η_i is $\chi(0,0) = \mathbf{Triv}_{W_{\mathbb{R}}}$ or $\chi(0,1) = \mathbf{sgn}_{W_{\mathbb{R}}}$, a'_i odd,
- $\delta_{t_j} = \delta(t_j/2, -t_j/2)$, $t_j + a_j - 1$ even.

the bad parity part can be written as $\psi_{bp} = \tau \oplus \check{\tau}$.

Arthur packets of $\mathbf{Sp}(2n,\mathbb{R})$: reduction to good parity

ψ : Arthur parameter for $G=\mathbf{Sp}(2n,\mathbb{R})$, $\psi=\psi_{gp}\oplus\psi_{bp}=\psi_{gp}\oplus\tau\oplus\check{\tau}$.

Problems : 1- Determine Langlands parameters of reps $\pi\in\Pi(G,\psi)$.

2- Show multiplicity one property. Then ρ_π is in $\widehat{A(\psi)}$.

3. Determine $\rho_\pi\in\widehat{A(\psi)}$.

Arthur packets of $\mathbf{Sp}(2n,\mathbb{R})$: reduction to good parity

ψ : Arthur parameter for $G = \mathbf{Sp}(2n,\mathbb{R})$, $\psi = \psi_{gp} \oplus \psi_{bp} = \psi_{gp} \oplus \tau \oplus \check{\tau}$.

$\tau : W_{\mathbb{R}} \times \mathbf{SL}_2 \to \mathbf{GL}(N_\tau,\mathbb{C})$ is the Arthur Langlands parameter of a well-determined irreducible unitary representation (irreducibly induced from Speh representations) of $\mathbf{GL}(N_\tau,\mathbb{R})$, that we denote again by τ.

ψ_{gp} is an Arthur parameter of good parity for a group $G_{gp} = \mathbf{Sp}(2(n-N_\tau),\mathbb{R})$.

$\mathbf{GL}(N_\tau,\mathbb{R}) \times \mathbf{Sp}(2(n-N_\tau),\mathbb{R})$ is the Levi subgroup of a parabolic $P \subset G$.

Assume we have constructed $\pi^A_{G_{gp}}(\psi_{gp})$. Then we can construct $\pi^A_G(\psi)$ by parabolic induction.

Fact : $A(\psi) \simeq A(\psi_{gp})$ canonically.

Arthur packets of $\mathbf{Sp}(2n,\mathbb{R})$: reduction to good parity

Theorem

$$\pi^A_G(\psi) = \bigoplus_{\eta\in\widehat{A(\psi)}} \mathrm{Ind}_P^G(\tau\boxtimes\pi^A_{G_{bp}}(\psi_{gp},\eta))\boxtimes\eta$$

If $\pi_{gp}\in\Pi(G_{gp},\psi_{gp})$, then $\mathrm{Ind}_P^G(\tau\boxtimes\pi_{gp})$ is irreducible.

Corollary

- *Multiplicity one property of $\Pi(G_{gp},\psi_{gp})$ implies multiplicity one for $\Pi(G,\psi)$.*
- *If Langlands parameters of elements of $\Pi(G_{gp},\psi_{gp})$ are known, so are the Langlands parameters for elements in $\Pi(G,\psi)$.*

Unipotent packets

We have reduced problems 1,2,3 to the case $\psi = \psi_{gp}$. Now suppose that $\psi = \psi_u$ (unipotent parameters).

Elements in $\Pi(G, \psi)$ are obtained by Howe's correspondence (symplectic/even orthogonal dual pairs).
Langlands parameters are notoriously hard to compute (ask specialists !).

Theorem

ψ unipotent, $\Pi(G, \psi)$ satisfy the multiplicity one property (Moeglin).
Elements in $\Pi(G, \psi)$ are weakly unipotent in the sense of Vogan

ρ_π's are not known (in general)

Reduction to unipotent case

$\psi = \psi_{gp} = \psi_u \oplus \psi_d$,

$$\psi_d = \bigoplus_{j=1}^{s} \delta_{t_j} \boxtimes R[a_j].$$

Order so that $t_j \geq t_{j+1}$ and if $t_j = t_{j+1}$ then $a_j \geq a_{j+1}$ (Fair range condition)

Notice that ψ_u is not always a parameter for a smaller symplectic group, ψ_u has value in $\mathbf{O}(2n_u + 1, \mathbb{C})$, where $2n_u + 1 = \dim \psi_u$.

$\psi_u' = \mathbf{sgn}_{W_\mathbb{R}}^{\dim \psi_d/2} \otimes \psi_u$ has value in $\mathbf{SO}(2n_u + 1, \mathbb{C})$ and is a parameter for $G_u = \mathbf{Sp}(2n_u, \mathbb{R})$.

Assume we know $\pi^A_{G_u}(\psi_u')$, we will describe $\pi^A_G(\psi)$ using Vogan-Zuckerman cohomological induction.

Reduction to unipotent case

$A(\psi) \simeq \{\pm 1\}^s \times A(\psi_u')$ generically (a quotient of the RHS in general).
So $\widehat{A(\psi)} \simeq \{\pm 1\}^s \times \widehat{A(\psi_u)}$, so write an element $\eta \in \widehat{A(\psi)}$ as
$\eta = (\eta_1, \ldots, \eta_s, \eta_u)$, with $\eta_i \in \{\pm 1\}$ and $\eta_u \in \widehat{A(\psi_u')}$.

Reduction to unipotent case

Set $\mathcal{Q} := \{\underline{q} = (q_1, \ldots, q_s) |\ q_j \in \{0, \ldots, a_j\}\}$.

Each $\underline{q} \in \mathcal{Q}$ determines a Levi subgroup $L_{\underline{q}}$ of G with

$$L_{\underline{q}} \simeq \prod_{j=1}^{s} \mathbf{U}(a_j - q_j, q_j) \times G_u$$

$\mathbb{L}_{\underline{q}}$ is the Levi factor of a θ-stable parabolic $\mathbb{Q}$ in $\mathbb{G}$ (setting for cohomological induction).

We will induce from $L_{\underline{q}}$ to G reps which are : unitary characters on $\mathbf{U}(a_j - q_j, q_j)$ factors (given by their inf. char since unitary groups are connected).

On the G_u factor, we put representations in $\Pi(G_u, \psi'_u)$, so weakly unipotent

This condition and the fair range condition implies that cohomological induction functors $\mathcal{R}^{\bullet}_{\mathfrak{q}, L_{\underline{q}}, G}$ vanish in all but one degree, let say $d_{\underline{q}}$. Set $\mathcal{R}_{\underline{q}} = \mathcal{R}^{d_{\underline{q}}}_{\mathfrak{q}, L_{\underline{q}}, G}$.

Reduction to unipotent case

Let us define $S_{\underline{q}} = (\epsilon_1, \dots, \epsilon_s) \in \{\pm 1\}^s$.

Set $a_{<j} = \sum_{i<j} a_i$ and $\beta_j = q_j(a_{<j}+1) + (a_j - q_j)a_{<j} + \frac{a_j(a_j+1)}{2}$, $\epsilon_j = (-1)^{\beta_j}$.
We define also unitary characters ξ_{t_j,q_j} of $\mathbf{U}(a_j - q_j, q_j)$, what precisely character to take is dictated by inf. char. consideration.

Theorem

$$\pi_G^A(\psi) = \sum_{\underline{q}\in\mathcal{Q}} S_{\underline{q}} \boxtimes \mathcal{R}_{\underline{q}}\left(\boxtimes_{j=1}^s \xi_{t_j,q_j} \boxtimes \pi_{G_u}^A(\psi_u')\right)$$

or equivalently, if $\eta \in \widehat{A(\psi)} \leftrightarrow (S_{\underline{q}}, \eta_u) \in \{\pm 1\}^s \times \widehat{A(\psi_u')}$,

$$\pi(G,\psi,\eta) = \mathcal{R}_{\underline{q}}\left(\boxtimes_{j=1}^s \xi_{t_j,q_j} \boxtimes \pi(G_u, \psi_u', \eta_u)\right)$$

Reduction to unipotent case

Why the formula for $S_{\underline{q}}$ so complicated ?

We need to choose Whittaker datum on G and G_u for the formula to make sense. So we need to choose Whittaker datum for all $\mathbf{Sp}(2N, \mathbb{R})$, $N \in \mathbb{Z}_{>0}$, simultaneously and in a compatible way.

"compatible" could mean "with respect to parabolic induction". That's usually what people do.

"compatible" could mean "with respect to cohomological induction".

One cannot have both....

Also, cohomological induction doesn't commute with endoscopic transfer. This induces complicated twists in the formula.

Regular case

Assume that $t_1 >> t_2 >> \cdots >> t_s >> 0$ (one can give precise bound)

Then cohomological induction is in the "good range"

This is good for our problems : one can compute Langlands parameters of elements in $\Pi(G, \psi)$ from Langlands parameters of elements in $\Pi(G_u, \psi'_u)$.

Multiplicity one is preserved.

Singular case

Assume only $t_1 \geq t_2 \geq \cdots \geq t_s$

Then cohomological induction is in the "fair range"

This is not so good for our problems (in fact very bad) : one cannot compute Langlands parameters of elements in $\Pi(G, \psi)$ from Langlands parameters of elements in $\Pi(G_u, \psi'_u)$ (in general).

Irreducibility is not preserved by cohomological induction. There can also be vanishing.

Multiplicity one is lost !

But no counter-exemple is known. There is still some hope to prove it.

Highest weight unitary modules

$G = \mathbf{Sp}(2n, \mathbb{R})$, θ : Cartan involution,
$K = G^\theta$ maximal compact subgroup, isomorphic to $\mathbf{U}(n)$.

$\mathfrak{g} = \mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$, $\mathfrak{g} = \mathfrak{k} \overset{\theta}{\oplus} \mathfrak{p} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$

$\mathfrak{t} \subset \mathfrak{k}$ Cartan subalgebra. Fix roots systems and compatible positive roots systems

$$R(\mathfrak{g}, \mathfrak{t}),\ R(\mathfrak{k}, \mathfrak{t}),\ R(\mathfrak{g}, \mathfrak{t})^+,\ R(\mathfrak{k}, \mathfrak{t})^+.$$

Identify $\mathfrak{t}$ and $\mathfrak{t}^*$ to $\mathbb{C}^n$ by choosing bases given by simple roots.

Highest weight unitary modules

Highest weight modules : (π, V) irreducible Harish-Chandra module for G such that there exists an highest weight vector $0 \neq v \in V$, $\mathfrak{n} \cdot v = 0$, where $\mathfrak{n}$ is the nilradical of a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$.

Fact. Either (exclusive cases) :

- V is finite dimensional, one can then take any $\mathfrak{b}$,
- V is infinite dimensional, and $\mathfrak{b} = \mathfrak{b}_{\mathfrak{k}} \oplus \mathfrak{p}^+$ (say V is holomorphic),
- V is infinite dimensional, and $\mathfrak{b} = \mathfrak{b}_{\mathfrak{k}} \oplus \mathfrak{p}^-$ (say V is antiholomorphic),

where $\mathfrak{b}_{\mathfrak{k}}$ is a Borel subalgebra of $\mathfrak{k}$.

if (π, V) is holomorphic, its contragredient is antiholomorphic. So we consider only holomorphic modules.

Highest weight unitary modules

Let $\mu = (m_1, \ldots, m_n) \in \mathbb{C}^n \simeq \mathfrak{t}^*$, $m_1 \geq m_2 \cdots \geq m_n$ integers.
$(\delta_{\mu^*}, V_{\mu^*}) \in \widehat{K}$ of highest weight $(-m_n, \ldots, -m_1)$.
$u =$ number of m_i equal to m_n, $v =$ number of m_i equal to $m_n + 1$.
The following is due to Kashiwara-Vergne and Enright-Howe-Wallach.

Theorem

Suppose $m_n \geq n - (u + v/2)$. Then there exists a unique irreducible unitary holomorphic module $(\pi(\mu), W_\mu)$ containing $(\delta_{\mu^*}, V_{\mu^*})$ with multiplicity one (so write $V_{\mu^*} \subset W_\mu$) such that the highest weight vector of V_{μ^*} is highest weight in W_μ.
All irreducible unitary holomorphic modules are obtained in this way.

If $\mu = (m, \ldots, m)$ (scalar case), we set $\pi_n(m) = \pi(\mu)$.

Description by Howe's correspondence

The $\pi(\mu)$ are obtained by Howe's correspondence with even orthogonal groups $\mathbf{O}(0, 2\ell)$ (the correspondence depends on the class of the real quadratic space, and not only on the isomorphism class of its orthogonal group, so we distinghish $\mathbf{O}(p, q)$ from $\mathbf{O}(q, p)$. There is also a choice of an additive character of $\mathbb{R}$ and we choose $x \mapsto \exp(i2\pi x)$).

An irreducible representation of $\mathbf{O}(0, 2\ell)$ restrict to $\mathbf{SO}(0, 2\ell)$ as :

$\bullet$ the sum of two irreducilble reps with respective highest weight $(\nu_1, \ldots, \nu_\ell)$ and $(\nu_1, \ldots, -\nu_\ell)$, $\nu_1 \geq \cdots \geq \nu_\ell \geq 0$. Let us denote it by $[\nu_1, \ldots, \nu_\ell]_+$.
$\bullet$ is irreducible with highest weight $(\nu_1, \ldots, \nu_\ell = 0)$ $\nu_1 \geq \cdots \geq \nu_\ell \geq 0$. There are two such reps, let us denote them by $[\nu_1, \ldots, \nu_\ell]_\pm$.

Description by Howe's correspondence

Theorem

Let $\mu = (m_1, \ldots, m_n)$. Then $\pi(\mu)$ is obtained by the Howe's correspondence from the following representations τ of $\mathbf{O}(0, 2\ell)$:

a) If $m_n > n$, then $\ell = n$ and $\tau = [m_1 - \ell, \ldots m_n - \ell]_+$,

b) If $m_n = n - a = \ell$, $0 \leq a \leq u$, then $\tau = [m_1 - \ell, \ldots, m_\ell - \ell]_+$,

c) If $m_n = n - u - b = \ell$, $2 \leq 2b \leq v$, then $\tau = [m_1 - \ell, \ldots, m_{n-u-2b} - \ell, \underbrace{0, \ldots, 0}_{b}]_-$,

d) If $m_n = n - a = \ell + 1$, $0 \leq a \leq u$, and $m_n \geq n + 1 - u/2$, then $\tau = [m_1 - \ell, \ldots, m_{2\ell-n} - \ell, \underbrace{0, \ldots, 0}_{n-2\ell}]_-$,

Arthur parameters for highest weight modules

Fix $\mu = (m_1, \dots, m_n)$ as above. See an infinitesimal character for G as an element of $\mathfrak{t}^* \simeq \mathbb{C}^n$ (up to conjugacy by the Weyl group). The infinitesimal character of $\pi(\mu)$ is integral, given by $(m_1 - 1, m_2 - 2, \dots m_n - n)$.

Assume that ψ is an Arthur parameter for G, with the same inf. char as $\pi(\mu)$, then

$$\psi = \psi_{gp} = \psi_u \oplus \psi_d = \bigoplus_{i=1}^{r} \eta_i \boxtimes R[a'_i] \oplus \bigoplus_{j=1}^{s} \delta_{t_j} \boxtimes R[a_j]$$

$r = 1$ or 3 and if $r = 3$, $a'_3 = 1$.

There is at most one j such that $t_j - a_j + 1 \leq 0$. If so $r = 1$, and if $t_j - a_j + 1 < 0$, then $a'_1 = 1$.

We set $a(\psi_u) = \max_i(a'_i)$. We assume the Fair Range condition.

Unipotent parameters and highest weight modules

We assume that $\psi = \psi_u$ is a unipotent Arthur parameter and $\Pi(G, \psi)$ has inf. char. as some unitary holomorphic module $\pi(\mu)$.

With the notation as above

$$\psi = \psi_u = \bigoplus_{i=1}^{r} \eta_i \boxtimes R[a_i']$$

with $r = 1$ or 3.

If $r = 1$ then $\psi = \mathbf{Triv}_{W_{\mathbb{R}}} \otimes R[2n+1]$; this is the parameter of trivial representation, which is the only element in the packet, so $\mu = (0, \ldots, 0)$ and $\pi(\mu)$ is the trivial representation.

Unipotent parameters and highest weight modules

If $r = 3$, write

$$\psi = \psi_u = (\eta_1 \boxtimes R[a]) \oplus (\eta_2 \boxtimes R[b]) \oplus (\eta_3 \boxtimes R[1])$$

with $a = a(\psi_u) \geq b \geq 1$ and $\eta_1\eta_2\eta_3 = \mathbf{Triv}_{W_{\mathbb{R}}}$.

Assume that $\eta_1 = \mathbf{sgn}_{W_{\mathbb{R}}}^{\frac{b+1}{2}}$, and assume that $\pi(\mu)$ is obtained by Howe's correspondence from a representation τ of $\mathbf{O}(0, b+1)$ in the Arthur packet with parameter

$$(\eta_2 \boxtimes R[b]) \oplus (\eta_3 \boxtimes R[1]).$$

Then, from inf. char. consideration, one see that τ is the trivial or determinant representation of $\mathbf{O}(0, b+1)$.

Unipotent parameters and highest weight modules

In the first case, $\mu = \left(\frac{b+1}{2}, \dots, \frac{b+1}{2}\right)$ and in the second case

$$\mu = \left(\underbrace{\frac{b+3}{2}, \dots, \frac{b+3}{2}}_{b+1}, \underbrace{\frac{b+1}{2}, \dots, \frac{b+1}{2}}_{n-b-1}\right)$$

so $n \geq b+1$. Let us denote in that case $\sigma_{n, \frac{b+1}{2}} = \pi(\mu)$.

Theorem

With notation as above

$$\psi = (\mathbf{sgn}_{W_\mathbb{R}}^{\frac{b+1}{2}} \boxtimes R[a]) \oplus (\eta_3 \mathbf{sgn}_{W_\mathbb{R}}^{\frac{b+1}{2}} \boxtimes R[b]) \oplus (\eta_3 \boxtimes R[1])$$

contains $\pi(\mu)$.

Regular infinitesimal character

The proof is global. It uses Yamana's results on global theta correspondence.

Let us fix now a regular inf. char. for $G = \mathbf{Sp}(2n, \mathbb{R})$, that is

$$\chi = (\chi_1, \dots, \chi_n), \quad \chi_i \in \mathbb{Z},\ \chi_1 > \dots > \chi_n > 0.$$

Suppose $\pi(\mu)$ has inf. char χ. Then either

- $m_n > n$. In this case

$$(\chi_1, \dots, \chi_n) = (m_1 - 1, m_2 - 2, \dots, m_n - n)$$

and $\pi(\mu)$ is an holomorphic discrete series.

Regular infinitesimal character

- $m_n = n - u = \ell$,

In the first case, set $a = 0$, in the second case, $a = u$.

χ being fixed, μ is determined by a integer $a \in \{0, \ldots, a_{max}\}$ where a_{max} is determined as follows

If $\chi_n \neq 1$, $a_{max} = 0$, if not a_{max} is the length k of the longest sequence

$$(\chi_{n-k+1}, \ldots, \chi_{n-1}, \chi_n) = (k, \ldots, 2, 1).$$

write then π_a for $\pi(\mu)$ with μ determined by a.

Regular infinitesimal character

Theorem

Let ψ be an Arthur parameter for G with regular integral inf. char χ as above. Let a_{max} obtained from χ as above. Then $\psi_u = \eta \boxtimes R[a(\psi_u)]$ with $a(\psi_u) \leq a_{max}$ and $\pi_a \in \Pi(G, \psi)$ iff $a(\psi_u) = 2a + 1$.

Such a parameter is called an Adams-Johnson parameter. The module $\pi(\mu) = \pi_a$ are $A_{\mathfrak{q}}(\lambda)$-modules (cohomologically induced from characters).

Scalar case

We now consider the scalar case $\mu = (m, \dots, m)$. We have already seen the regular case, so we assume $0 \leq m \leq n$. The inf. char of $\pi_n(m)$ is given by $(m-1, \dots, m-n)$.

Theorem

Assume that ψ is an Arthur parameter such that $\pi_n(m) \in \Pi(G, \psi)$. Then we are in one of the following cases :
(i) $\dim \psi_u = 1$, $2m+1 > n+1$, $\psi = \eta \boxtimes R[1] \oplus \bigoplus_{j=1}^{s} \delta_{t_j} \boxtimes R[a_j]$ *and for* $j = 1, \dots s-1$,

$$[t_j - a_j + 1, t_j + a_j - 1] \cap [t_{j+1-}a_{j+1} + 1, t_{j+1} + a_{j+1} - 1] = \emptyset$$

(ii) $\psi = (\mathbf{sgn}_{W_{\mathbb{R}}}^{\frac{2n+1-a(\psi_u)}{2}} \boxtimes R[a(\psi_u)]) \oplus \psi'$,
ψ' *is a parameter for* $\mathbf{O}(0, 2n+1-a(\psi_u))$ *such that* $\Pi(\mathbf{O}(0, 2n+1-a(\psi_u), \psi')$ *contains a finite dimensional irred. rep.* $E_{\psi'}$ *which corresponds to* $\pi_n(m)$ *in Howe's correpondence.*
The multiplicity of $\pi_n(m)$ *in* $\Pi(G, \psi)$ *is one.*

Scalar case

Theorem

In case (ii), either :
(iia) $a(\psi_u) = 2(n-m)+1$, *and* $E_{\psi'}$ *is the trivial rep. of* $\mathbf{O}(0,2m)$,
(iib) or $2m \geq n+2$, $a(\psi_u) = 2(n-m)+3$, *and* $E_{\psi'}$ *is the rep.* $[\underbrace{1,\dots 1}_{2(m-1)-n}, \underbrace{0,\dots,0}_{n-m+1}]_-$ *of* $\mathbf{O}(0,2(m-1))$.
$\pi_n(m)$ *is cohomologically induced from a Levi subgroup*

$$L \simeq \prod_j \mathbf{U}(0,a_j) \times \mathbf{Sp}(\dim \psi_u - 1)$$

from a representation which is the appropriate character on the unitary factors, and
$\pi_{n_u}(m_u)$, $n_u = (\dim \psi_u - 1)/2$, $m_u = (\dim(\psi_u) - a(\psi_u))/2$ *in case (iia),*
σ_{n_u,k_u}, $k_u = (\dim(\psi_u) - a(\psi_u))/2$ *in case (iib),*
on the $\mathbf{Sp}(\dim \psi_u - 1)$ *factor.*

Recall the unipotent reps $\sigma_{n,k} = \pi(\mu)$, $2k \geq n-1$, $\mu = (\underbrace{k+1,\dots,k+1}_{2k},\underbrace{k,\dots,k}_{n-2k})$ $(\sigma_{2k,k} = \pi_{2k}(k+1))$.

Theorem

Let ψ be an Arthur parameter with same inf. char. as $\sigma_{n,k}$. Then $\pi \in \Pi(G,\psi)$ iff and only if $a(\psi_u) = 2(n-k)+1$ and $\mathbf{sgn}^k_{W_{\mathbb{R}}} \boxtimes R[2(n-k)+1]$ occurs in ψ.
The multiplicity of π in $\Pi(G,\psi)$ is one.

Langlands parameters of $\pi_n(m)$ and $\sigma_{n,k}$

Theorem

- *$\pi_n(m)$ is the Langlands quotient of*

$$\mathrm{Ind}_P^G(\mathbf{sgn}^m|.|^{n-m} \boxtimes \cdots \boxtimes \mathbf{sgn}^m|.| \boxtimes \pi_m(m))$$

where $P = MN$, $M \simeq (\mathbb{R}^\times)^{n-m} \times \mathbf{Sp}(2m, \mathbb{R})$.
($\pi_m(m)$ is a limit of holomorphic discrete series for $\mathbf{Sp}(2m, \mathbb{R})$).

- *$\sigma_{n,k}$ is the Langlands quotient of*

$$\mathrm{Ind}_P^G(\mathbf{sgn}^k|.|^{n-k} \boxtimes \cdots \boxtimes \mathbf{sgn}^k|.|^{k+1} \boxtimes \mathbf{sgn}^k|.|^{k-1} \boxtimes \mathbf{sgn}^k|.| \boxtimes \pi_{k+1}(k+1))$$

where $P = MN$, $M \simeq (\mathbb{R}^\times)^{n-k-1} \times \mathbf{Sp}(2(k+1), \mathbb{R})$.

Computation of ρ_π for $\pi = \pi_n(m)$ or $\pi = \sigma_{n,m}$

We now know that for $\pi = \pi_n(m)$ or $\pi = \sigma_{n,m}$ (or their contragredient) the multiplicity of π in any Arthur packet containing it is one.

The remaining problem is now to compute the character ρ_π of $A(\psi)$ attached to π and ψ.

The characterization of $\pi^A(\psi)$ by endoscopic identities in Arthur 2013 depends on the normalization of the so called *transfer factors* (Langlands-Shelstad and Kottwitz-Shelstad), i.e. on the choice of a Whittaker datum for G.

For $G = \mathbf{Sp}(2n, \mathbb{R})$ there are two inequivalent choices that we distinguish by a choice of $\delta \in \{\pm 1\}$.

Computation of ρ_π for $\pi = \pi_n(m)$ or $\pi = \sigma_{n,m}$

We start with unipotent parameters. Assume

$$\psi = (\mathbf{Triv}_{W_\mathbb{R}} \boxtimes R[1]) \oplus (\mathbf{sgn}^m_{W_\mathbb{R}} \boxtimes R[2m-1]) \oplus (\mathbf{sgn}^m_{W_\mathbb{R}} \boxtimes R[2(n-m)+1])$$

$A(\psi) = S(\mathbf{O}(1) \times \mathbf{O}(1) \times \mathbf{O}(1)) \simeq \{(\epsilon_1, \epsilon_2, \epsilon_3) \in (\mathbb{Z}/2\mathbb{Z})^3,\ \epsilon_1\epsilon_2\epsilon_3 = 1\}$.
$\widehat{A(\psi)} = (\mathbb{Z}/2\mathbb{Z})^3/\sim$.

$$\begin{aligned}
\pi &= \pi_n(m) & \rho_\pi &= (1, (-1)^{\lfloor \frac{-\delta m}{2} \rfloor}, (-1)^{\lfloor \frac{-\delta m}{2} \rfloor}) \\
\pi &= \sigma_{n,m} & \rho_\pi &= (-1, (-1)^{1+\lfloor \frac{-\delta m}{2} \rfloor}, (-1)^{\lfloor \frac{-\delta m}{2} \rfloor}) \\
\pi &= \pi_n(m)^* & \rho_\pi &= (1, (-1)^{\lfloor \frac{\delta m}{2} \rfloor}, (-1)^{\lfloor \frac{\delta m}{2} \rfloor}) \\
\pi &= \sigma^*_{n,m} & \rho_\pi &= (-1, (-1)^{1+\lfloor \frac{\delta m}{2} \rfloor}, (-1)^{\lfloor \frac{\delta m}{2} \rfloor})
\end{aligned}$$

Computation of ρ_π for $\pi = \pi_n(m)$ or $\pi = \sigma_{n,m}$

Assume now

$$\psi = (\mathbf{sgn}_{W_\mathbb{R}} \boxtimes R[1]) \oplus (\mathbf{sgn}_{W_\mathbb{R}}^{m+1} \boxtimes R[2m-1]) \oplus (\mathbf{sgn}_{W_\mathbb{R}}^{m} \boxtimes R[2(n-m)+1])$$

same $A(\psi)$.

$$\pi = \pi_n(m) \qquad \rho_\pi = ((-1)^{\frac{1+\delta}{2}+m}, (-1)^{\frac{1+\delta}{2}+m}(-1)^{\lfloor \frac{-\delta m}{2} \rfloor}, (-1)^{\lfloor \frac{-\delta m}{2} \rfloor})$$

$$\pi = \sigma_{n,m} \qquad \rho_\pi = ((-1)^{\frac{1+\delta}{2}+m+1}, (-1)^{\frac{1+\delta}{2}+m+1}(-1)^{\lfloor \frac{-\delta m}{2} \rfloor}, (-1)^{\lfloor \frac{-\delta m}{2} \rfloor})$$

$$\pi = \pi_n(m)^* \qquad \rho_\pi = ((-1)^{\frac{1+\delta}{2}+m}, (-1)^{\frac{1+\delta}{2}+m}(-1)^{\lfloor \frac{\delta m}{2} \rfloor}, (-1)^{\lfloor \frac{\delta m}{2} \rfloor})$$

$$\pi = \sigma_{n,m}^* \qquad \rho_\pi = ((-1)^{\frac{1+\delta}{2}+m+1}, (-1)^{\frac{1+\delta}{2}+m+1}(-1)^{\lfloor \frac{\delta m}{2} \rfloor}, (-1)^{\lfloor \frac{\delta m}{2} \rfloor})$$

Computation of ρ_π for $\pi = \pi_n(m)$ or $\pi = \sigma_{n,m}$

Assume now

$$\psi = \psi_u \oplus \psi_d = \psi_u \oplus \bigoplus_{j=1}^{s} \delta_{t_j} \boxtimes R[a_j]$$

and that $\pi_n(m)^* \in \Pi(G, \psi)$.

$A(\psi)$ is identified with functions ϵ from the set of block, of ψ to $\{\pm 1\}$ with same value on the same block, modulo a global sign change, so in the generic case

$$\widehat{A(\psi)} = (\mathbb{Z}/2\mathbb{Z})^s \times \widehat{A(\psi_u)}$$

Let us distinguish 3 cases :
(1) ψ_u is irreducible,
(2) ψ_u contains 3 blocks, one of them being $\mathbf{sgn}_{W_{\mathbb{R}}}^{m} \boxtimes R[2(n-m)+1]$
(3) ψ_u contains 3 blocks, one of them being $\mathbf{sgn}_{W_{\mathbb{R}}}^{m+1} \boxtimes R[2(n-m)+3]$

Computation of ρ_π for $\pi = \pi_n(m)$ or $\pi = \sigma_{n,m}$

In case (2) and (3) write the other two blocs as $(\eta_1 \boxtimes R[1]) \oplus (\eta_2 \boxtimes R[2a-1])$.
Set $a_{<i} = \sum_{j<i} a_j$, $\delta_i = \delta(-1)^{a_{<i}}$, $\delta' = \delta(-1)^{\sum_{i=1}^s a_j}$.

For $\pi \in \Pi(G,\psi)$ with multiplicity one, $\rho_\pi \in \widehat{A(\psi)}$ is is given by a function ϵ on the blocks of ψ, so by $\epsilon(\delta_{t_j} \boxtimes R[a_j])$, $j = 1, \dots s$, and in case (2) and (3) by a triplet $(\epsilon_1, \epsilon_2, \epsilon_3)$ modulo global sign, so it is enough to give $\epsilon_1\epsilon_2$ and $\epsilon_2\epsilon_3$).

Computation of ρ_π for $\pi = \pi_n(m)$ or $\pi = \sigma_{n,m}$

For $\pi = \pi_n(m)^*$, ϵ corresponding to ρ_π is given by

for $j = 1, \dots s$, $\epsilon(\delta_{t_i} \boxtimes R[a_i]) = (-1)^{\lfloor \frac{\delta_i a_i}{2} \rfloor}$,

$\epsilon_1 \epsilon_2 = (-1)^{\lfloor \frac{\delta' a}{2} \rfloor}$,

In case (2) $\epsilon_2 \epsilon_3 = \begin{cases} 1 \text{ if } \eta_2 = \mathbf{sgn}_{W_\mathbb{R}}^m \\ \delta'(-1)^{a+1} \text{ if } \eta_2 = \mathbf{sgn}_{W_\mathbb{R}}^{m+1} \end{cases}$

In case (3) $\epsilon_2 \epsilon_3 = \begin{cases} -1 \text{ if } \eta_2 = \mathbf{sgn}_{W_\mathbb{R}}^{m-1} \\ \delta'(-1)^{a} \text{ if } \eta_2 = \mathbf{sgn}_{W_\mathbb{R}}^{m} \end{cases}$