Introduction to Hilbert modular varieties

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Introduction

This is the enhanced version talk note of the summer school 2005 at Hakui in Ishikawa Prefecture. I gave two talks, one hour for each. The former is mainly about the cohomology groups of Hilbert modular varieties, the latter about the interpretation as moduli spaces of abelian varieties with real multiplication. Hence the contents of talks are of basic level. What might be the hope and intension of the organizers and the audience, in two hours the possible thing is very limited.

There are still many problems for Hilbert modular varieties and Hilbert modular forms. I hope this might be of some help who are interested in this theme.

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1 Introduction of Hilbert modular varieties

One way to introduce Hilbert modular varieties is to regard them as moduli spaces of abelian varieties with maximal real multiplication. Namely let F be a totally real number field of degree $g = [F : \mathbf{Q}]$, $\mathfrak{o}$ an order of F, then we consider the moduli space of abelian varieties A of dimension g with a ring homomorphism $\theta : \mathfrak{o} \to \mathrm{End}(A)$. For simplicity as the order $\mathfrak{o}$ of F, we consider only the integer ring O_F of F.

Let A be defined over the complex number field $\mathbf{C}$, then its tangent space at the identity $V = \mathrm{Lie}(A)$ is canonically an $O_F \otimes \mathbf{C}$-module, which is automatically of rank 1 by the condition of the dimension of A. If we denote by $P_\infty(F)$ the set of equivalence classes of the infinite places of F, we have the canonical decomposition:

$$O_F \otimes_{\mathbf{Z}} \mathbf{C} = \bigoplus_{v \in P_\infty(F)} \mathbf{C} e_v.$$

Here e_v is the primitive idempotent of $O_F \otimes \mathbf{C}$ corresponding to the place v. By definition we have $1 = \sum_v e_v$, and we have the associated decomposition of V:

$$V = \bigoplus_{v \in P_\infty(F)} V_v \qquad (V_v = e_v V, \quad \text{for each } v \in P_\infty(F)).$$

To have A as a complex torus, we have to specify its lattice L such that $A = V/L$. Since the rank of L over $\mathbf{Z}$ is $2g$, it should be a projective O_F-module of rank 2. As known in Algebraic Number Theory, such L is isomorphic to a direct sum $O_F \oplus \mathfrak{a}^{-1}$ with a fractional ideal $\mathfrak{a}^{-1}$ in F.

To have a complex tours V/L is equivalent to assure that

(i) there is an injective homomorphism $\varphi : L \to V$

(ii) with the quotient $V/\varphi(L)$ being compact.

In this case, we also say that L is a O_F-lattice of V.

Definition 1.1 For a fixed L and V We set

$$X^* := \mathrm{Hom}_{O_F}(L, V) := \{\varphi : L \to V \mid \varphi(L) \text{ is a lattice } \}.$$

Fix an injection L in F^2 once for all. This is equivalent to fix a marking $\alpha : L \otimes_{\mathbf{Z}} \mathbf{Q} \cong F^2$. We have to define also $\beta : V \cong O_F \otimes_{\mathbf{Z}} \mathbf{C}$. Under these markings, to specify φ is equivalent to specify the pair of elements $z = \varphi(e_1),\ w = \varphi(e_2)$ in $O_F \otimes_{\mathbf{Z}} \mathbf{C}$ with $e_1 = (1, 0),\ e_2 = (0, 1) \in F^2$. And the condition that $\varphi(L)$ is a lattice is equivalent to

$$\mathrm{Im}(z_v \bar{w}_v - \bar{z}_v w_v) \neq 0 \text{ for each } v \in P_\infty(F).$$

Thus we have to consider $\tilde{X}^*$ consisting of a pair of points $z,\ w$ satisfying the above condition.

We can write $\tilde{X}^*$ as a sum of 2^g connected components X_ε^*, such that each of them is consisting of the pair (z, w) with specified signatures

$$\varepsilon_v = \mathrm{sign}\{\mathrm{Im}(z_v \bar{w}_v - \bar{z}_v w_v)\} \in \{\pm\} \text{ for each } v \in P_\infty(F).$$

Here $\varepsilon = (\varepsilon_v)_{v \in P_\infty(F)}$ is a vector of signs belonging to $\mu_2^{P_\infty(F)}$ with $\mu_2 = \{\pm 1\}$.

To recover the original X^* from $\tilde{X}^*$, we have to 'forget' $\alpha,\ \beta$. To forget β is to consider the quotient of $\tilde{X}^*$ under the action of the unit group $(O_F \otimes \mathbf{C})^\times$ of $O_F \otimes \mathbf{C}$. It is easy to see that each component $\tilde{X}_\varepsilon^*$ is stable under this unit group. As an representation of each orbit under this unit group, we may choose an element of the form $(z, 1)$ in $\tilde{X}^*$. Thus we have a natural identification:

$$\tilde{X}^*/(O_F \otimes \mathbf{C})^\times \cong (\mathbf{C} - \mathbf{R})^g = \bigcup_\varepsilon \mathbf{H}_\varepsilon =: X_\infty$$

where each $\mathbf{H}_\varepsilon$ is given by

$$\mathbf{H}_\varepsilon = \{z = (z_v) \mid \mathrm{sign}(\mathrm{Im}(z_v)) = \varepsilon\} \cong X_\varepsilon^*/(O_F \otimes \mathbf{C})^\times.$$

Next we can forget the marking α i.e., the choice of special basis $e_1,\ e_2$ in F^2 or in $O_F \oplus \mathfrak{a}^{-1}$ by taking the quotient with respect to the action of $GL_{O_F}(O_F \oplus \mathfrak{a})$. Thus the double coset space

$$X^* \cong GL_{O_F}(O_F \oplus \mathfrak{a}) \backslash \tilde{X}^*/(O_F \otimes \mathbf{C})^\times = GL_{O_F}(L) \backslash X_\infty$$

is the moduli space of complex O_F-tori, the space parametrizing the isomorphism classes of the whole complex O_F tori.

In order to pass from complex tori to abelian varieties, we have to choose polarizations on tori if they exist. There exist many polarizations on each A. We have to specify them. This problem is discussed later in the section of "Polarization and weak Polarization." RIGHT???

From now on, we fix a subgroup Γ in $GL_2^+(F)$ commensurable with $GL_2^+(O_F)$, and form the associated quotient $V = V_{\Gamma,\varepsilon} = \Gamma \backslash \mathbf{H}_\varepsilon$. In the first part of this note, we recall the basic results on the cohomology groups of this variety.

2 Part A: Hilbert modular variety $V_{\Gamma,\varepsilon}$ as a complex algebraic variety

2.1 Singularities inside

The group Γ acts on $\mathbf{H}_\varepsilon$ properly discontinuously. The elements in the center $Z(\Gamma) = \Gamma \cap Z(GL_2^+(O_F))$ acts on $\mathbf{H}_\varepsilon$ trivially. For a given point z in $\mathbf{H}_\varepsilon$, the stablizer Γ_z in Γ is equal to $\Gamma \cap G^+_{\mathbf{R},z}$. Here $G_{\mathbf{R},z}$ is the stabiliser of z in $G_{\mathbf{R}} = GL(2, F \otimes_{\mathbf{Q}} \mathbf{R})$ is isomorphic to a compact abelian group $SO(2)^g$ modulo center $Z(G^+_{\mathbf{R}})$ ($G^+_{\mathbf{R}}$ =the identity component of $GL_2^+(F \otimes_{\mathbf{Q}} \mathbf{R})$). Therefore $\Gamma_z/Z(\Gamma)$ is a finite abelian group. Hence the singularities of the complex analytic space V is at most quotient singularities by finite abelian groups. There are only finite number of such singularities modulo Γ. The conjugacy classes in Γ represented by non-central elements in some Γ_z are called *elliptic* conjugacy classes.

Remark The elliptic singularities are so mild that even at such point x the local cohomology group with rational coefficients $H^*_{\{x\}}(V, \mathbf{Q}) := H^*(V, \bmod V - \{x\}, \mathbf{Q})$ satisfies the axiom of Poincaré, hence we have Poincare duality theorem with rational coefficients. As such V is said to be *rationally smooth*.

You can find detailed descriptions of examples for this kind of singularities in the cases of surfaces, if you consult with the books of Hirzebruch [16] or van der Geer [17].

2.2 Algebraicity

Theorem 2.1 *The quotient V is a quasi-projective variety over $\boldsymbol{C}$.*

Proof. The first essential result is due to Siegel, who showed that the field of meromorphic functions on V has transcendental degree g over $\mathbf{C}$. This implies that V is an algebraic variety. Afterward, the compactification of such varieties are considered (i.e., Baily-Borel, Satake compactification, which is now refered as the *minimal compactification* sometimes). These compactification are mapped to a projective space, by linear system spanned automorphic forms including Eisenstein series. The image in the projective space has finite number of singularities corresponding to cusps.

Remark Now Siegel's result is an immediate consequence of the dimension formula of modular forms.

2.3 Cusps and the minimal compactification

The space $\mathbf{H}_\varepsilon$ has the rational boundary component $P^1(F) = F \cup \{\infty\}$ with respect to $GL_2(F)$, which is the homogeneous space. The set of cusps is the double coset space $\Gamma\backslash P^1(F) = \Gamma\backslash GL_2(F)/B_F$, which is a finite set by Reduction Theory of linear algebraic groups over number fields (*cf.* [Borel], [Platonov]).

The minimal compactification V^* of V is the union $V \cup \Gamma\backslash P^1(F)$ as a set. We have to define a natural topology and the natural structure of normal analytic space around these newly attached finite number of points. Each cusp c is mapped to the cusp ∞ by an element δ in $SL(2, F)$. Therefore, replacing $SL(2, O_F)$ by its transform $\Gamma' := \delta^{-1} SL(2, O_F)\delta$, we may regard the cusp is ∞. Then its stabilizer of ∞ in $SL(2, O_F)$, the Borel subgroup B_F and the unipotent radical N_F are given by

$$B_F := \left\{ \begin{pmatrix} \varepsilon & m \\ 0 & \varepsilon^{-1} \end{pmatrix} \mid m \in F,\ \varepsilon \in F^\times \right\}, \qquad N_F := \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \mid m \in F \right\},$$

The intersections $\Gamma' \cap N_F$ and $\Gamma' \cap B_F$ are of the form:

$$\left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \mid m \in M_c \right\}, \quad \left\{ \begin{pmatrix} \varepsilon & m \\ 0 & \varepsilon^{-1} \end{pmatrix} \mid m \in M_c,\ \varepsilon \in V_c \right\}$$

respectively, with a fractional ideal M_c in F and a subgroup V_c of finite index in the unit group $O_F^\times$ of O_F.

The space V^* is a compact complex analytic variety with cusps singularities along cusps.

2.4 Toroidal compactifications

The toroidal compactification is the method to attach divisors at infinity to compactify V^*. This is defined, depending on the optional data on r.p.p decomposition associated to the unipotent radical N_c of the parabolic subgroup P_c associated with the cusp c.

Resolution of cusp singularities of Hilbert modular varieties might be the easiest examples of toroidal compactifications.

In the case of surfaces ($g = 2$), there is a detailed description of resolution obtained by continued fractions.

See the paper of Ehlers [15] for general n.

3 Part B: Cohomology groups of Hilbert modular varieties

We review the basic results on the cohomology groups $H^*(V, \mathbf{C})$ for $V = V_{\Gamma,\varepsilon}$. In the elliptic modular case, there is the isomorphism of Eichler-Shimura. For higher dimensional case, this was generalized by Yozo Matsushima for cocompact case. A. Borel had been the leading person to extend this kind results to non-cocompact case.

In the Hilbert modular case, it suffices to review some old result of Matsushima-Shimura [6], and for non-cocompact case the result of Harder and the speaker of this talk.

3.1 Matsushima-Shimura isomorphism (cocompact case)

The cohomology groups consists of two parts: one is the universal part "not depending on Γ" and the "essential" part described in terms of Hilbert modular forms. Before discussing the Hilbert modular case where Γ has cusps, hence V is non-compact, we firstly consider the case when Γ is a cocompact discrete subgroup of $G = SL(2, \mathbf{R})^n \times G_0$ (G_0 a compact group, normally a product of $SU(2)$'s). We assume that Γ is *irreducible* throughout in this section, i.e., the projection. to any factor $SL(2, \mathbf{R})^m$ ($m < n$) of G has dense image.

3.1.1 *The finite dimensional representations of G and the associated sheaf*

Any complex irreducible representation ρ of G with finite dimension is of the form $\rho = \rho' \otimes \sigma$, with ρ' a finite dimensional irreducible representation of $SL(2, \mathbf{R})^n$, and σ an irreducible continuous representation of the compact group G_0. Moreover $\rho' \cong \otimes_{i=1}^n \mathrm{Sym}^{k_i}$ with each Sym^{k_i} the symmetric tensor representation of degree k_i

$$\mathrm{Sym}^{k_i} : SL_2(\mathbf{R}) \to GL_{k_i+1}(\mathbf{C}).$$

We denote the pull-back of this representation to the subgroup Γ by $E = E_\rho$. When Γ is torsion-free, it is a representation of the fundamental group $\pi_1(V, *) \cong \Gamma$ of E, hence corresponds to a local system $\tilde{E} = \tilde{E}_\rho$. When Γ has torsion elements, we firstly assume that -1_2 acts trivially on E. And moreover it has elliptic fixed points on V, we can define firstly a local system corresponding to E on $V - \{\text{elliptic fixed points}\}$, and after that take the direct image j_* to the whole V by the inclusion immersion into V, to get a constructible sheaf $\tilde{E}$ on V corresponding to E.

We put $P_\infty = \{1, \cdots, n\}$ and let $\varepsilon \in \mathrm{Map}(P_\infty, \{\pm\}) = \{\pm\}^{P_\infty}$. Then we put $\mathbf{H}_\varepsilon = \prod_{v \in P_\infty} H_{\varepsilon(v)}$.

Lemma 3.1 *Assume that Γ has a torsion-free (normal) subgroup Γ' of finite index. Then we have the canonical isomorphism $H^i(\Gamma, E_\rho) \cong H^i(V, \tilde{E}_\rho)$.*

Proof) If Γ itself is torsion-free, $V = \Gamma' \backslash \mathbf{H}_\varepsilon$ is a $K(\pi, 1)$ space, hence Lemma is true. In general. we can apply the spectral sequences

$$E_2^{p,q} = H^p(\Gamma/\Gamma', H^q(\Gamma', E)) \Rightarrow H^{p+q}(\Gamma, E), \quad E_2^{p,q} = H^p(\Gamma/\Gamma', H^q(V', E)) \Rightarrow H^{p+q}(V, \tilde{E})$$

This settle the proof. One may refer to Grothendieck Tohoku.

We denote by W_σ the representation space of σ and by W_σ^Γ the subspace consisting of invariant vectors under Γ in W_σ.

3.1.2 A digression to $P^1(\mathbf{C})$

Let (z_0, z_1) be coordinates on $\mathbf{C}^2$, and denote by $\pi : \mathbf{C}^2 - \{0\} \to P^1(\mathbf{C})$ the standard projection map to construct the projective line. Let $U \subset P^1(\mathbf{C})$ be an open set and $Z : U \to \mathbf{C}^2 - \{0\}$ a lifting of U, i.e., a holomorphic map with $\pi \circ Z = \mathrm{id}$. Consider the differential form

$$\omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \|Z\|^2.$$

If $Z' : U \to \mathbf{C}^2 - \{0\}$ is another lifting, then $Z' = f \cdot Z$ with f a non-zero holomorphic function, so that

$$\begin{aligned}
\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \|Z'\|^2 &= \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} (\log \|Z\|^2 + \log f + \log \bar{f}) \\
&= \omega + \frac{\sqrt{-1}}{2\pi} (\partial\bar{\partial} \log f - \bar{\partial}\partial \log \bar{f}) \\
&= \omega
\end{aligned}$$

is globally defined differential form on $P^1(\mathbf{C})$. It is of type $(1,1)$. With respect to the natural action of $U(2)$ on $P^1(\mathbf{C})$, it is invariant. This means that if ω is positive at one point, iff it is positive everywhere.

Set $w_1 = z_1/z_0$ on $U = \{z_0 \neq 0\}$, then $Z = (1, w_1)$ is a lifting of U. Then

$$\begin{aligned}
\omega &= \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log(1 + w_1 \bar{w}_1) \\
&= \frac{\sqrt{-1}}{2\pi} \partial \left(\frac{w_1 d\bar{w}_1}{1 + |w_1|^2} \right) \\
&= \frac{\sqrt{-1}}{2\pi} \left[\frac{w_1 \bar{\partial} dw_1}{1 + |w_1|^2} - \frac{\bar{w}_1 dw_1 \wedge w_1 d\bar{w}_1}{(1 + |w_1|^2)^2} \right]
\end{aligned}$$

and at the point $[1:0]$, $\omega = \frac{\sqrt{-1}}{2\pi} dw_1 \wedge d\bar{w}_1 > 0$.

On the non-compact dual H, i.e., the complex upper half plane, the metric form $\kappa = \frac{d\tau \wedge d\bar{\tau}}{\mathrm{Im}\,\tau^2}$ for the $SL(2, \mathbf{R})$-invariant metric $ds^2 = |d\tau|^2 / (\mathrm{Im}\,\tau)^2$. It is of type $(1,1)$ and d-closed real form, i.e., Kählerian.

3.1.3 *The universal cohomology classes*

On $P^1(\mathbf{C})$, the Fubini-Study metric defines the associated $(1,1)$-type form which is the Kähler form as we see in the previous subsubsection. On its non-compact dual H the complex upper (or lower) half space, we can consider the $(1,1)$-type form $\kappa = \frac{d\tau \wedge d\bar{\tau}}{\mathrm{Im}(\tau)^2}$ which is invariant under the action of either $SL_2(\mathbf{R})$ or $GL_2^+(\mathbf{R})$.

Definition 3.1 If $(k_1, k_2, \cdots, k_n) \neq 0$, i.e., ρ' is not trivial, we put $H^i_{univ}(V, E) = \{0\}$ for any i. If $(k_1, \cdots, k_n) = 0$, we set

$$H^i_{univ}(V, \tilde{E}) := \begin{cases} \{0\} & \text{if } i \neq 2p \\ \{\bigoplus_{P \subset P_\infty, \# P = p} \boldsymbol{C}\kappa_P\} \otimes \{W_\sigma\}^\Gamma & \text{if } i = 2p. \end{cases}$$

Here for each subset P of P_∞ the (p,p)-type form, given by

$$\kappa_P = \bigwedge_{v \in P} \kappa_v$$

is descent to the quotient V to define a closed form on V, with each κ_v is the Kähler form on each half complex plane $H_{\varepsilon(v)}$.

When Γ is cocompact (hence not a Hilbert modular group), the above group is found to be a subspace of $H^i(V, \tilde{E})$, independent of the choice of the cocompact Γ. Similar fact is also true for Hilbert modular case, as discussed later.

3.1.4 *Modular forms and cusp forms*

There are a few different way to define modular forms on $\mathbf{H}_\varepsilon$. We give the most *down-to-earth (?)* definition here, though it is a bit awkward (?).

We firstly have to define automorphy factor. For each $v \in P_\infty$, we have to define another new parity η_v, *holomorphy* or *anti-holomorphy*. To give the parity function η on P_∞ is equivalent to define a partition $J_+ \cup J_- = P_\infty$. Here

$$J_\pm = \{v \in P_\infty \mid \eta_v = \pm\}.$$

Now for a given non-negative integer k_v and a parity η_v we define

$$j^{\eta_v}_{k_v}(g_v, z_v) = \begin{cases} \det(g_v)^{-k_v/2}(c_v z_v + d_v)^{k_v} & (\eta_v = +), \\ \det(g_v)^{-k_v/2}(c_v \bar{z}_v + d_v)^{k_v} & (\eta_v = -). \end{cases}$$

Here $g_v = \begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix}$, and $z_v \in H_{\varepsilon_v}$.

Here is the definition of the automorphy factor associated with the partition $J_+ \cup J_- = P_\infty(F)$.

Definition 3.2 For $\mathbf{k} = (k_v)_{v \in P_\infty} \in \boldsymbol{Z}_{\geq 0}$ we put

$$j^{(J_+, J_-)}_{\mathbf{k}}(g, z) = j^\eta_{\mathbf{k}}(g, z) := \prod_{v \in P_\infty(F)} j^\eta_{k_v}(g_v, z_v)$$

The automorphic forms with η-holomorphy are defined as follows.

Definition 3.3 *We call a function* $f : \mathbf{H}_\varepsilon \to \boldsymbol{C}$ *η-holomorphic, if it is holomorphic in* z_v *if* $\eta_v = +$ *and anti-holomorphic in* z_v *if* $\eta_v = -1$. *Set*

$$\begin{aligned} S_{\mathbf{k}}^{(J_+,J_-)}(\Gamma) = \quad & \{f : \mathbf{H}_\varepsilon \to \boldsymbol{C}, \eta\text{-holomorphic function} \\ & |\text{(i) } f(\gamma(z)) = j_{\mathbf{k}}^{(J_+,J_-)}(g,z)f(z) \text{ for any } \gamma \in \Gamma, \ \text{(ii) } f(z) \text{ is } 0 \text{ at each cusp } \}. \end{aligned}$$

Definition 3.4 *For* $\mathbf{k} = (k_1, \cdots, k_n)$ *we put*

$$H^i_{ess}(V, \tilde{E}_\rho) := \begin{cases} \{0\} & \text{if } i \neq n, \\ \bigoplus_{I_+ \cup I_- = P_\infty(F)} S_{\mathbf{k+2}}^{(I_+,I_-)}(\Gamma) \otimes \{W_\sigma\}^\Gamma & \text{if } i = n. \end{cases}$$

In particular, we have

$$H^i_{ess}(V, \boldsymbol{C}) = \begin{cases} \{0\} & \text{if } i \neq g, \\ \bigoplus_{I_+ \cup I_- = P_\infty(F)} S_{\mathbf{2}}^{(I_+,I_-)}(\Gamma) & \text{if } i = g. \end{cases}$$

Note here that there are 2^n *partitions* $I_+ \cup I_- = P_\infty$ *of* P_∞, $\mathbf{2} = (2, \cdots, 2)$, *and* $\mathbf{k} + \mathbf{2}$ *the addition of integral vectors.*

Theorem 3.1 (Matsushima-Shimura) *For* $E = E_\rho$ *we have*

$$H^i(V, \tilde{E}) = H^i_{univ}(V, \tilde{E}) \oplus H^i_{ess}(V, \tilde{E}).$$

Proof) This is now a very special case of the theory of $(\mathfrak{g}, K)$-cohomology theory. We may refer to Borel-Wallach [21]. As we have seen in Lemma (**), since $\mathbf{H}_\varepsilon$ is contractible to a point, we have $H^i(V, E_\rho) = H^i(\Gamma, \tilde{E}_\rho)$. Write $G' \cong SL(2, \mathbf{R})^n$ and $K' = SO(2)^n$, then $\mathbf{H}_\varepsilon \cong G'/K' = SL(2, \mathbf{R})^n/SO(2)^n$.

Then, because the space $L^2(\Gamma\backslash G)$ or the subspace consisting of C^∞ vectors on this space is regarded as a smooth induction of the Γ-module E, we have

$$H^i(\Gamma, E) \cong H^i_\infty(G, L^2(\Gamma\backslash G)_\infty \otimes_{\mathbf{C}} E).$$

Here $L^2(\Gamma\backslash G)_\infty$ is the subspace of smooth vectors in $L^2(\Gamma\backslash G)$), and $H^i_\infty(G, *)$ is the smooth cohomology for Lie groups G. One can pass to the $(\mathfrak{g}, K)$-cohomology

$$H^i(\mathfrak{g}, K; L^2(\Gamma\backslash G)_\infty \otimes E).$$

via van Est spectral sequence, with $\mathfrak{g} = \mathrm{Lie}(G)$ and K a maximal compact subgroup of G.

Remark 3.1. There is another way to get this isomorphism:

(Iso-1) $\qquad H^i(V, \tilde{E}) \cong H^i(\mathfrak{g}, K; L^2(\Gamma\backslash G)_\infty \otimes E).$

By de Rham theorem we have $H^i(V, \mathbf{C}) \cong H^i(\Omega^*(\mathbf{H}_\varepsilon)^\Gamma)$ with $\Omega^*(\mathbf{H}_\varepsilon)$ the de Rham cohomology on $\mathbf{H}_\varepsilon$. This was essentially the original approach by Matsushima.

Now we recall a basic result on the spectral decomposition.

Proposition 3.1 (Gelfand, Graev, Piatetski-Shapiro [23]) *Let* $\hat{G}$ *be the unitary dual of* G, *i.e., the set of unitary equivalence classes of irreducible unitary representations of* G. *Then as a unitary* G*-module with its right quasi-regular action,* $L^2(\Gamma\backslash G)$ *have a discrete decomposition into closed irreducible (unitary) representations* π *of* G

$$L^2(\Gamma\backslash G) = \widehat{\bigoplus_{\pi \in \hat{G}}} \{\mathrm{Hom}_G(H_\pi, L^2(\Gamma\backslash G))\} \otimes H_\pi = \widehat{\bigoplus_{\pi \in \hat{G}}} m_\Gamma(\pi) H_\pi \qquad (m_\Gamma(\pi) < \infty).$$

Here $\widehat{\bigoplus}$ *means the Hilbert space direct sum,* H_π *the representation space of* π, *and* $m_\Gamma(\pi)$ *the multiplicity of* π *in* $L^2(\Gamma\backslash G)$, *i.e.,* $m_\Gamma(\pi)$ *is the dimension of the intertwining space* $\mathrm{Hom}_G(H_\pi, L^2(\Gamma\backslash G))$ *consisting of bounded linear operators compatible with* G*-actions.*

Remark 3.2 In general $L^2(\Gamma\backslash G)$ is written as a direct integral of unitary irreducible representations, because G is a group of type I. The continuous spectrum is described in terms of Eisenstein series which are intertwiners between the continuous spectrum and the principal series representations of G.

In our situation, it is better to see the contents of the main objects more precisely. We have $G = G' \times G_0$ and the corresponding Lie algebras give $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}_0$, with compact factor G_0. Since G_0 is contained in K (K is of the form $K' \times G_0$ with a maximal compact K' in G'), we have

$$H^i(\mathfrak{g}, K; L^2(\Gamma\backslash G)\otimes E_\rho) \cong H^i(\mathfrak{g}', K'; L^2(\mathrm{pr}'(\Gamma)\backslash G')_\infty \otimes E_{\rho'}) \otimes H^0(\mathfrak{g}_0, G_0; L^2(\mathrm{pr}_0(\Gamma)\backslash G_0)\otimes W_\sigma).$$

Note that the right factor in the last tensor product of cohomology group is $(W_\sigma)^\Gamma$.

Thus applying Proposition 3.1 above, we have

$$H^i(V, \tilde{E}) \cong \bigoplus_{\pi \in \hat{G}'} \{\mathrm{Hom}_{(\mathfrak{g}',K')}(H_\pi, L^2(\mathrm{pr}'(\Gamma)\backslash G')) \otimes H^i(\mathfrak{g}', K'; H_\pi \otimes E_{\rho'})\} \otimes (W_\sigma)^\Gamma.$$

Note here that the topological sum $\widehat{\bigoplus}$ is now replaced by the algebraic sum $\bigoplus$ (this is a lemma by A. Borel, proved by using that $H^i(V, \tilde{E})$ is of finite dimension).

The investigation of $H^i(\mathfrak{g}', K'; H_\pi \otimes E_{\rho'})$ is a local problem, i.e., it does not depend on Γ, and $\mathfrak{g}'$, H_π and $E_{\rho'}$ decompose into simple factors. So the problem is reduced to the case of $\mathrm{Lie}(SL(2, \mathbf{R}))$ and its unitary representations. We have to enumerate those representations of $SL(2, \mathbf{R})$ which contribute to $H^i(\mathfrak{sl}(2, \mathbf{R}), SO(2); H_\pi \otimes \mathrm{Sym}^k)$. The discrete series representations $D^\pm_{k+2}$ (holomorphic and anti-holomorphic) with the Blattner parameter $k+2$ contribute to H^1 with cohomology dimension 1. If $k = 0$ the trivial representation 1 is the other representation contributing to H^0 and H^2 with cohomology groups $\mathbf{C}$. The global condition of the irreducibility of Γ is utilized to exclude the case when H_π is a tensor product of non-zero number of the trivial representations and non-zero numbers of the discrete series representations.

The intertwining space $\mathrm{Hom}_{(\mathfrak{g}',K')}(H_\pi, L^2(\Gamma'\backslash G'))$ is identified with the space of cusp forms $S^{(I_+,I_-)}_{\mathbf{k+2}}$ if $H_\pi = \otimes D^{\eta_\tau}_{k_\tau}$ with η is the signature distribution corresponding to the partition (I_+, I_-).

The proof presented here works for any semisimple Lie group G and a cocompact discrete subgroup Γ, and it is slightly different from the original proof. But the main idea behind is the same.

3.2 Hilbert modular case (non-cocompact case)

The quotient space $V := V_{\Gamma,\varepsilon} = \Gamma\backslash\mathbf{H}_\varepsilon$ is non-compact for congruence subgroups $\Gamma \subset SL_2(O_F)$. One may believe that the mixed structure on the cohomology groups of the open variety V have non-trivial arithmetic information. But before that we want to grasp the 'pure' part.

To get homogeneous Hodge structures, one way is to consider the intersection cohomology groups $IH^i(V^*, \mathbf{Q})$ of the minimal compactification V^* with *middle perversity*. The other natural way from the view point of the theory of harmonic integrals is to consider the L^2-cohomology. Note that V has the canonical Kähler metric induced from that of $\mathbf{H}_\varepsilon$. Then we can define a subcomplex of the de Rham complex $\{(\mathcal{A}^*(V),\ d)\}$ of V by

$$\mathcal{A}^i_{(2)}(V) := \{\omega \in \mathcal{A}^i(V) \mid \omega \text{ and } d\omega \text{ are } L^2\}.$$

Then the cohomology of this complex is the L^2 cohomology $H^i_{(2)}(V, \mathbf{C})$ and we have canonical homomorphisms

$$H^i_c(V, \mathbf{C}) \to H^i_{(2)}(V, \mathbf{C}) \to H^i(V, \mathbf{C}),$$

(H^i_c: a compact support cohomology). We define a symbol for the image of the second arrow: $H^i_{[2]}(V, \mathbf{C}) := \mathrm{Im}\{H^i_{(2)}(V, \mathbf{C}) \to H^i(V, \mathbf{C})\}$, and the image of the composition $H^i_!(V, \mathbf{C}) := \mathrm{Im}\{H^i_c(V, \mathbf{C}) \to H^i(V, \mathbf{C})\}$ is called the *interior* cohomology group.

A nice fact is $IH^*(V^*, \mathbf{C}) \cong H^*_{(2)}(V, \mathbf{C})$ and the Hodge filtrations are compatible. This isomorphism is a special case of a more general similar fact valid for arithmetic quotients of bounded symmetric domains, which was one time called 'Zucker conjecture' and later proved by Saper-Stern and Looijenga in '80's. Note that And $IH^*(V^*, \mathbf{Q})$ and $H^i_!(V, \mathbf{Q})$ have pure rational Hodge structure of weight i.

There is a result by Harder to decompose $H^i(V, \mathbf{C})$ as a direct sum $H^i(V, \mathbf{C}) = H^i_!(V, \mathbf{C}) \oplus H^i_{Eis}(V, \mathbf{C})$ of the interior cohomology $H^i_!(V, \mathbf{C})$ and *Eisenstein cohomology classes* $H^i_{Eis}(V, \mathbf{C})$ ([4, 5]).

L^2-cohomology groups are firstly investigated by A. Borel extensively and we can recover much of the 'old world' of the cocompact case, if we replace $H^i(V)$ by $H^i_{(2)}(V)$.

For simplicity we describe the result in the case of constant coefficients. We have a natural isomorphism:

$$H^*_{(2)}(V, \mathbf{C}) \cong H^*(\mathfrak{g}, K; L^2(\Gamma\backslash\tilde{G}/Z(\tilde{G}))),$$

with $\tilde{G} = GL_2^+(O_F \otimes \mathbf{R})$ and $\mathfrak{g} = ****$.

As shown generally by Borel-Casselman [24], there is no contribution from the continuous spectrum L^2_{cont} in the spectral decomposition :

$$L^2(\Gamma\backslash\bar{G}) = L^2_{dis}(\Gamma\backslash\bar{G}) \oplus L^2_{cont}(\Gamma\backslash\bar{G}),$$

where $\bar{G} = \tilde{G}/Z(\tilde{G})$ and $L^2_{dis}(\Gamma\backslash\bar{G}))$ is the sum of closed $\bar{G}$-invariant irreducible subspaces and $L^2_{cont}(\Gamma\backslash\bar{G})$ is its orthogonal complement by definition. The continuous part $L^2_{cont}(\Gamma\backslash\bar{G})$ is intertwined by Eisenstein series.

Anyway we have

$$H^*_{(2)}(V, \mathbf{C}) \cong H^*(\mathfrak{g}, K; L^2_{dis}(\Gamma\backslash\bar{G})).$$

A general theorem of Langlands tells that $L^2_{dis}(\Gamma\backslash\bar{G})$ consists of the cuspidal part and the residual part of the Eisenstein series :

$$L^2_{dis}(\Gamma\backslash\bar{G}) = L^2_{cusp}(\Gamma\backslash\bar{G}) \oplus L^2_{res}(\Gamma\backslash\bar{G}).$$

Here the submodules

$$H^*(\mathfrak{g}, K; L^2_{res}(\Gamma\backslash\bar{G}))$$

is the universal part $H^*_{(2),univ}(V, \mathbf{C})$ of $H^*_{(2)}(V, \mathbf{C})$ generated by the invariant Kähler classes as in the cocompact case. The other part is

$$H^*(\mathfrak{g}, K; L^2_{cusp}(\Gamma\backslash\bar{G})) = H^*_{(2),ess}(V, \mathbf{C}).$$

We can define the automorphy factor and the space of cusp forms similarly as in the cocompact case. Here we have to replace P_∞ by $P_\infty(F)$, and to define cusp forms we have to impose the vanishing condition at cusps. We have a theorem analogous to that of Matsushima-Shimura.

Here is the relation between various cohomology groups.

Theorem 3.2 *In the case of Hilbert modular varieties, the square-integrable cohomology, and the interior cohomology are coincide if the degree of the cohomology group is $\leq g$. They are sums of the universal cohomology classes and the cuspidal cohomology classes.*

There is a good survey in Chapter III of Freitag's book [14].

When the coefficients system is trivial, for toroidal compactification $\tilde{V}$, we can consider the canonical map:

$$H^g(\tilde{V}, \mathbf{Q}) \to H^g(V, \mathbf{Q}).$$

In [8], we proved that the image of this natural map is $W_g H^g(V, \mathbf{Q})$.

3.3 The action of Hecke operators

Let $\Delta \subset \Gamma$ be a subgroup of finite index in Γ, and let $p : V_\Delta := \Delta\backslash\mathbf{H}_\varepsilon \to V_\Gamma = \Gamma\backslash\mathbf{H}_\varepsilon$ be the associated finite morphism of analytic spaces. Then the direct image $p_*\mathbf{Q}$ defines a constructible sheaf on V_Γ. The natural maps $\mathbf{Q} \hookrightarrow p_*\mathbf{Q}$ and $tr : p_*\mathbf{Q} \to \mathbf{Q}$ induces

$$\begin{array}{ll} p^* : H^i(V_\Gamma, \mathbf{Q}) \to H^i(V_\Delta, \mathbf{Q}), & p_* : H^i(V_\Delta, \mathbf{Q}) \to H^i(V_\Gamma, \mathbf{Q}), \\ p_c^* : H_c^i(V_\Gamma, \mathbf{Q}) \to H_c^i(V_\Delta, \mathbf{Q}), & p_{*,c} : H_c^i(V_\Delta, \mathbf{Q}) \to H_c^i(V_\Gamma, \mathbf{Q}), \end{array}$$

because congruence subgroups $\Gamma,\ \Delta$ we have compatible minimal compactifications to define cohomology groups with compact supports: $H_c^i(V_\Gamma, \mathbf{Q}) := H^i(V_\Gamma, i_!\mathbf{Q})$ $(i : V_\Gamma \subset V_\Gamma^*$. Similarly, we have canonical extensions to the intermediate extensions $i_{!*}\mathbf{Q}$ and $i_{!*}p_*\mathbf{Q}$ to V^* to get

$$p_{!*}^* : IH^i(V_\Gamma, \mathbf{Q}) \to IH^i(V_\Delta, \mathbf{Q}), \quad p_{*,!*} : IH^i(V_\Delta, \mathbf{Q}) \to IH^i(V_\Gamma, \mathbf{Q}).$$

Here p^* and $p_{c,*}$ are mutual Poincaré dual, etc.

If we have $\Delta = \Gamma \cap \alpha\Gamma\alpha^{-1}$ for some element in the commensurator of Γ, we have two finite morphisms

$$\begin{array}{ccccc} & & V_\Delta & & \\ & p \swarrow & & q \searrow & \\ V_\Gamma & & & & V_{\alpha\Gamma\alpha^{-1}} \cong V_\Gamma. \end{array}$$

Then we have a composition $q_{*,\heartsuit} \circ p_\heartsuit^* : H_\heartsuit^i(V_\Gamma, \mathbf{Q}) \to H_\heartsuit^i(V_\Gamma, \mathbf{Q})$ $\ (\heartsuit \in \{\text{empty},\ c, !*\})$.

Applying this for $T(\mathfrak{p})$ or $T(\mathfrak{n})$ operators of Hecke, we have actions of these operators on $H_\heartsuit^i(V_{K_{fin}}, \mathbf{Q})$ as endomorphisms of rational mixed Hodge structures. Here $V_{K_{fin}} = \cup_c V_{\Gamma_c}$ considered in the next subsection.

3.4 Hodge structures attached to primitive Hilbert modular forms

To have reasonable action of Hecke operators, we have to replace V_Γ by a finite disjoint sum of such V_Γ:

$$\mathcal{V}_{K_f} := GL_2(F)\backslash X_\infty \times GL_2(\mathbf{A}_{fin})/K_{fin}.$$

Recall here that $X_\infty = (\mathbf{C} - \mathbf{R})^g$. Moreover, here $\mathbf{A}_{fin}$ is the finite adeles of F and K_{fin} is a compact open subgroup of that, corresponding to a 'congruence subgroup.' Recall the approximation theorem, to write $\mathcal{V}_{K_{fin}}$ as a finite disjoint sum $\bigcup_c V_{\Gamma_c}$ with $V_{\Gamma_c} = \Gamma_c\backslash\mathbf{H}_{\varepsilon_c}$. Then the sum $H_{ess}^g(\mathcal{V}_{K_{fin}}, \mathbf{Q}) = \bigoplus_c H_{ess}^g(V_{\Gamma_c}, \mathbf{Q})$ of the essential part of the interior cohomology group $H_!^g(V_{\Gamma_c}, \mathbf{Q})$ of degree g is stable under the action of Hecke operators. The ring of Hecke operators acts as a commutative subring R in $\mathrm{End}(H_{ess}^g(\mathcal{V}_{K_{fin}}, \mathbf{Q}))$ and as a semisimple algebra because of the existence of polarization. Therefore the $\mathbf{Q}$-algebra R is a direct sum of finite separable extensions K of $\mathbf{Q}$.

By extension of scalars, these fields K are known to be the fields of eigenvalues of Hecke operators of some Hecke eigenform in $S_2^\eta(K_{fin}) = S_2^{(I_+, I_-)}(K_{fin}) := \bigoplus_c S_2^{(I_+, I_-)}(\Gamma_c)$ for each signature distribution $\eta \in \mathrm{Map}(P_\infty(F), \mu_2)$ corresponding to the partition $I_+ \cup I_- = P_\infty(F)$.

In the case when Γ or K_{fin} is not full modular case, we have to consider primitive forms. applying the theory of new forms. etc...

Then the new part $H^g_{!,new}(\mathcal{V}_{K_{fin}}, \mathbf{Q})$ is a polarized Hodge structure of weight g, which is a Hecke submodule of rank 2^g, and the image R_{new} of the of degree g is a direct sum of finite separable extension. Let e be a primitive idempotent of R_{new}, then $K_e = eR_{new}e = e \cdot R_{new}$ is an algebraic number field and the rational sub-Hodge structure

$$H^g(M_f, \mathbf{Q}) := eH^g_{!,new}(\mathcal{V}_{K_{fin}}, \mathbf{Q})$$

which is a K_e module of rank 2^g. For each embedding $\sigma : K_e \to \mathbf{C}$, we have an associated primitive form $f_\sigma^{I_+,I_-}$ corresponding to the partition $I_+ \cup I_- = P_\infty(F)$, such that whose eigenvalue $C(\mathfrak{p})$ at $\mathfrak{p}$ is the image $\sigma(t(\mathfrak{p}))$. Here $t(\mathfrak{p})$ is the image of $T(\mathfrak{p})$ in $R_{new} \to K_e$. We have $\sigma(K_e) = K_{f^\sigma}$, the field generated by eigenvalues of f_σ over $\mathbf{Q}$.

3.4.1 The Frobenius at infinity

We refer [8] for this subsection and the next.

In order to define a fundamental system of generators in the Betti cohomology group, one has to define 'Frobenius at infinity' corresponding to the signature distributions on $P_\infty(F)$.

Firstly on

$$X_\infty = (\mathbf{C} - \mathbf{R})^{P_\infty(F)} = \coprod_{\varepsilon \in \mathrm{Map}(P_\infty(F), \{\pm 1\})} \mathbf{H}_\varepsilon,$$

an involution F_v is defined by

$$\tilde{F}_v : \tau = (\tau_w)_{w \in P_\infty(F)} \in X_\infty \mapsto \tau' = (\tau'_v) \text{ with } \tau'_w = \begin{cases} \tau_w, & \text{if } w \neq v, \\ \bar{\tau}_w, & \text{if } w = v. \end{cases}$$

Obviously $\tilde{F}_v^2 = \mathrm{id}$ and this passes through the quotient by any $\Gamma \subset GL_2(O_F)$. Therefore we have induced actions F_v of these g involutions on the cohomology group $H^i(\Gamma \backslash X_\infty, \mathbf{Q})$. This defines a finite abelian group $\mathcal{F}$ of $(2, \cdots, 2)$ type of order 2^g.

For each $\eta \in \mathrm{Map}(P_\infty(F), \{\pm 1\})$, we can consider the associate character of $\mathcal{F}$ and define η-eigenspace $H^i(\Gamma \backslash X_\infty, \mathbf{Q})_\eta$. On the w-th Kähler class $[\kappa]$, we have $F_v([\kappa_w]) = (-1)^{\delta_{v,w}}[\kappa_w]$ with $\delta_{v,w}$ the Kronecker delta. Obviously we have the decomposition:

$$H^g(\Gamma \backslash X_\infty, \mathbf{Q}) = \bigoplus_{\eta \in \hat{\mathcal{F}}} H^g(\Gamma \backslash X_\infty, \mathbf{Q})_\eta$$

Moreover since the action of Hecke operators are commutative with that of $\mathcal{F}$, we have also the induced decomposition:

$$H^g(M_f, \mathbf{Q}) = \bigoplus_{\eta \in \hat{\mathcal{F}}} H^g(M_f, \mathbf{Q})_\eta$$

with $H^g(M_f, \mathbf{Q})_\eta = H^g(M_f, \mathbf{Q}) \cap H^g(\Gamma \backslash X_\infty, \mathbf{Q})_\eta$. Then each $H^g(M_f, \mathbf{Q})_\eta$ is a K_f module of rank 1. Here we denote K_{e_f} by K_f for the idempotent e_f corresponding to f.

3.5 The Hodge structures with marking attached to primitive forms

Now we can choose a system of generators $\{\gamma_\eta\}_\eta$ such that each $\gamma_\eta \in H^g(M_f, \mathbf{Q})_\eta$ and

$$\langle \gamma_\eta, \gamma_{\eta'} \rangle = \delta_{\eta, \eta'}.$$

Here $\langle *, * \rangle$ is the polarization form on $H^g(M_f, \mathbf{Q})$ and $\delta_{\eta,\eta'}$ is the Kronecker delta. We may consider this a canonical basis of the Betti group $H^g(M_f, \mathbf{Q})$.

On the other hand, the de Rham realization of $H^g(M_f, \mathbf{Q}) \otimes \mathbf{C} = H^g(M_f, \mathbf{C})$ is the sum

$$\bigoplus_{\eta \in \mathrm{Map}(P_\infty(F), \mu_2)} \{S_2^\eta(K_f) \cap H^g(M_f, \mathbf{C})\}.$$

Here each $S_2^\eta(M_f) := S_2^\eta(K_f) \cap H^g(M_f, \mathbf{C})$ is a free $K_f \otimes \mathbf{C}$ of rank 1. We can choose a primitive form f_σ^η corresponding to the embedding $\sigma : K_f \hookrightarrow \mathbf{C}$ for each fixed η. Then we have a basis $B^\eta = \{f_\sigma^\eta | \ \sigma \in \mathrm{Emb}(K_f, \mathbf{C})\}$ of $S_2^\eta(M_f)$ for each η. And $\bigcup_\eta B^\eta$ is a basis of the whole $H^g(M_f, \mathbf{C})$.

Choose one $\sigma : K_f \hookrightarrow \mathbf{C}$, and consider the corresponding subspace $H^g(M_f, \mathbf{Q}) \otimes_{K_f, \sigma} \mathbf{C}$ in $H^g(M_f, \mathbf{C})$, which is identified with $\bigoplus_{\eta \in \mathrm{Map}(P_\infty(F), \mu_2)} \mathbf{C} f_\sigma^\eta$. Then via the period map which connects the de Rham realization and the Betti realization, we have a system of complex numbers $\{c_\gamma(f_\sigma^\eta)\}$ such that $f_\sigma^\eta = \sum_\gamma c_\gamma(f_\sigma^\eta)\gamma^{(\sigma)}$. Here $\gamma^{(\sigma)}$ is the canonical image of each γ with respect to $H^g(M_f, \mathbf{Q}) \to H^g(M_f, \mathbf{Q}) \otimes_{K_f, \sigma} \mathbf{C}$.

These $2^g[K_f, \mathbf{Q}]$ numbers are *the fundamental system of periods with respect to the canonical basis* $\{\gamma\}$ *and the choice of basis* $\{B^\eta \mid \eta \in \mathrm{Map}(P_\infty(F), \mu_2)\}$. We have the Riemann-Hodge period relation for them (*cf.* [8]).

4 Part C: Hilbert modular varieties as moduli spaces

We can give a description of Hilbert modular varieties as moduli spaces of abelian varieties with maximal real multiplication. This is the analogy to the elliptic modular curves which are moduli spaces of elliptic curves with adequate level structures.

4.1 The elliptic modular case

Each elliptic curve has the canonical polarization coming from the unit element o. The three times $3o$ of the divisor o is very ample and define an immersion $|3o| : E \hookrightarrow \mathbf{P}^2$ as a smooth cubic curve in the projective plane. Choose a non-trivial section x in $\mathcal{O}_E(2o)$ which determined up to affine transformation $ax + b$ and another section y of $\mathcal{O}_E(3o)$ which is outside of $\mathcal{O}_E(2o)$, after renormalization of $x,\ y$, we have a equation of E:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2a_4x + a_6$$

with a_i the defining coefficients of E. Then we can define $g_2,\ g_3,\ \Delta$ of E and the invariant (i.e., the algebraic modulus) $j(E) = 12^3g_2^3/\Delta$. Here is the classical result.

Proposition 4.1 *Let E_i $(i = 1,\ 2)$ be two elliptic curves over $\boldsymbol{C}$. Then E_1 and E_2 are isomorphic, iff $j(E_1) = j(E_2)$.*

Here is an obvious corollary of the above proposition. Let $\mathrm{Aut}(\mathbf{C}/\mathbf{Q}) = \mathrm{Aut}(\mathbf{C})$ be the group of isomorphisms of the field $\mathbf{C}$. Note here that any automorphism σ of $\mathbf{C}$ induces the identity map on the prime field $\mathbf{Q}$. For a given elliptic curve, we consider a subgroup

$$\mathrm{Stabl}(E) := \{\sigma \in \mathrm{Aut}(\mathbf{C}) \mid E \cong E^\sigma\}.$$

Then we have $j(E) = j(E^\sigma) = j(E)^\sigma$ for any $\sigma \in \mathrm{Stabl}(E)$. This means the fixed subfield $\mathbf{C}^{\mathrm{Stabl}(E)}$ of $\mathrm{Stabl}(E)$ in $\mathbf{C}$ contains the field of moduli $\mathbf{Q}(j(E))$, and the converse inclusion is also easy to prove. Then we have $\mathbf{Q}(j(E)) = \mathbf{C}^{\mathrm{Stabl}(E)}$. In the case of elliptic curve, we can say more: there is an elliptic curve E_0 defined over $\mathbf{Q}(j(E))$ with the same j-invariant.

Generalizing this to polarized abelian varieties $(A, \mathcal{C})$ (for definition, see §§4.2), we can define the field of moduli as follows. Put

$$\mathrm{Stabl}(A, \mathcal{C}) := \{\sigma \in \mathrm{Aut}(\mathbf{C}) \mid (A, \mathcal{C}) \cong (A^\sigma, \mathcal{C}^\sigma)\}.$$

Then the field of moduli $k_{m(A,\mathcal{C})}$ is defined as the fixed subfield

$$k_{m(A,\mathcal{C})} := \mathbf{C}^{\mathrm{Stabl}(A,\mathcal{C})}.$$

If one can find a set of algebraic invariants $J_1(A, \mathcal{C}), \cdots, J_N(A, \mathcal{C})$ such that $(A_1, \mathcal{C}_1) \cong (A_2, \mathcal{C}_2)$ iff $J_i(A_1, \mathcal{C}_1) = J_i(A_2, \mathcal{C}_2)$ $(1 \leq i \leq N)$, then $k_{m(A,\mathcal{C})} = \mathbf{Q}(J_1(A, \mathcal{C}), \cdots, J_N(A, \mathcal{C}))$.

In this field-theoretic approach, which is the style of the classical papers of Shimura, the construction of $J_i(A, \mathcal{C})$ are done by using the Chow form or the Chow coordinates [25] of

$$|m\mathcal{C}| : A \hookrightarrow \mathbf{P}^N \qquad (m \text{ large enough}).$$

The regular structure of the moduli space is given over $\mathbf{C}$ via transcendental construction of moduli space using the periods or the integral Hodge structures of $(A, \mathcal{C})$. In this case, it is the quotient $Sp(g, \mathbf{Z}) \backslash \mathbf{H}_g$ of the Siegel upper half space by the Siegel modular group. We can show the algebraicity of this kind quotient space by Siegel, or Satake, Bailey-Borel compactification.

4.2 Divisors and Picard groups on abelian varieties

Any known method of algebraic construction of the moduli spaces uses some kind of 'projective geometry.' We have to consider the parametrized varieties as subvarieties in a fixed projective space. This means that we have to choose a polarization of our abelian varieties.

Definition 4.1 Given an abelian variety A over a field k, a polarization $\mathcal{C}$ on A is an algebraic equivalence class $[D]$ of an ample divisor D on A.

For a given (Weil) divisor D on A, we can associate an invertible sheaf or a line bundle $\mathcal{O}_A(D)$. Then two linearly equivalent divisors D_1, D_2 gives isomorphic invertible sheaves. Conversely, given an invertible sheaf $\mathcal{L}$, then it is trivial over the rational function field of A, hence there is a divisor D such that $L \cong \mathcal{O}_A(D)$. In particular the notion of Weil divisor up to linear equivalence is equivalent to the notion of Cartier divisor. Thus the Picard group $\mathrm{Pic}(A)$ is defined in two ways:

$$\begin{aligned} \mathrm{Pic}(A) &:= \{D \text{ divisors on } A\}/\underset{\text{linear equivalence}}{\sim} \\ &:= \{L \text{ invertible sheave on } A\}/\underset{\text{isomorphism}}{\sim}. \end{aligned}$$

Definition 4.2 Two divisors D_1 and D_2 over A is said to be algebraically equivalent, if there is a parameter space S with two closed points s_1, s_2 on S together with a divisor $\mathcal{D} \subset A \times S$ such that $D_i = \pi^{-1}(s_i)$ $(i = 1, 2)$ for π the composition of the inclusion $D \subset A \times S$ and the second projection pr_2.

Definition 4.3 Let $\mathrm{Pic}^0(A)$ be the subgroup of $\mathrm{Pic}(A)$ consisting of divisors algebraically equivalent to 0. Then the quotient group $\mathrm{NS}(A) := \mathrm{Pic}(A)/\mathrm{Pic}^0(A)$ is called the Neron-Severi group of A.

A divisor D on A is called *very ample*, if the associated linear system defines a closed immersion

$$|D| : A \hookrightarrow \mathbf{P}^N.$$

The same notion is said *ample* in the old literature. A divisor D is called *ample* if some positive multiple mD of D is very ample.

Proposition 4.2 *The set*

$$\mathrm{NS}^+(A) := \{[D] \mid D \text{ is ample}\}$$

is a cone.

Let $(Schemes)/k$ be the category of schemes over k. For a variable scheme S, we associate the group

$$\mathrm{Pic}_{A/k}(S) := \mathrm{Pic}(A \times S)/\mathrm{Pic}(S).$$

Then this defines a contravariant functor $\mathrm{Pic}_{A/k}$ from $(Schemes)/k$ to the category of abelian groups (Ab). As shown by Grothendieck and Raynaud, there exists a locally noetherian group scheme $\mathbf{Pic}_{A/k}$ representing this functor. Let $\mathbf{Pic}^0_{A/k}$ be the connected component of $\mathbf{Pic}_{A/k}$. Then it is the dual abelian variety A^* of A. The quotient group $\mathbf{Pic}_{A/k}/\mathbf{Pic}^0_{A/k}$ is canonically isomorphic to the Neron-Severi group $\mathrm{NS}(A)$ taking the k-valued points of $\mathbf{Pic}_{A/k}$.

If a divisor D or an invertible sheaf L is given on A, we can define a homomorphism $\varphi_D : A \to A^*$ or $\varphi_L : A \to A^*$ by

$$x \in A(\bar{k}) \mapsto Tx^*(D) - D \text{ or } T_x^* L \otimes L^{-1} \in \mathrm{Pic}^0(A) = A^*(\bar{k})$$

for geometric points x in A. Since φ_D is additive (i.e., $\varphi_{D_1+D_2} = \varphi_{D_1} + \varphi_{D_2}$) is the 0-homomorphism for $D \in \mathrm{Pic}^0(A)$, we have a naturally induced homomorphism

$$\mathrm{NS}(A) \to \mathrm{Hom}(A, A^*).$$

Moreover the image of this homomorphism is contained in the fixed part $\mathrm{Hom}(A, A^*)^{sym}$.

Example When $\dim(A/\mathbf{C}) = 2$, we have $\mathrm{rank}_{\mathbf{Z}}\mathrm{Hom}(A, A^*)^{sym} \leq 3$. When this is equal to 3, there are infinite many different F with O_F with $\theta : O_F \to \mathrm{End}(A)$.

If the base field k is the complex number field, the notion of algebraic equivalence is equivalent to the notion of the homological equivalence. i.e., the first Chern classes of two divisors in $H^2(A^{an}, \mathbf{Z})$ coincides. Therefore in this case, we have a canonical homomorphism:

$$c_1 : \mathrm{NS}(A) \to H^2(A^{an}, \mathbf{Z}) = \bigwedge^2 H^1(A^{an}, \mathbf{Z}).$$

Here A^{an} is the analytic variety canonically associated with the algebraic variety $A/\mathbf{C}$.

Definition 4.4 A pair $(A, \mathcal{C})$ *of an abelian variety* A *over* k *and a class* $\mathcal{C} = [D] \in \mathrm{NS}^+(A)$ *is called a polarized abelian variety.*

Remark Since $\mathcal{C}$ is ample, there is a positive integer m such that $|m\mathcal{C}| : A \hookrightarrow \mathbf{P}^N$ is a immersion. And if two polarizations $\mathcal{C}$, $\mathcal{C}'$ belongs to the same $\mathbf{Q}_+$-ray, i.e., there are positive integers m, m' such that $m\mathcal{C} = m'\mathcal{C}'$, then via Veronese maps the two embedding are related, i.e., there are relations between the coefficients of the two systems of defining equations of A in $\mathbf{P}^N$ or $\mathbf{P}^{N'}$. But if $\mathcal{C}$ and $\mathcal{C}'$ does not belongs the same $\mathbf{Q}_+$-ray, probably the projective equations for A are quite different. For a generic abelian variety A, we have $\mathrm{NS}(A) \cong \mathbf{Z}$. Hence $\mathrm{NS}^+(A) \cong \mathbf{N}$ consists of unique $\mathbf{Q}_+$-ray.

Now let us assume that A is defined over $\mathbf{C}$ such that A^{an} is a O_F-torus. A given polarization class $\mathcal{C} = [D] \in NS^+(A) \subset H^2(A^{an}, \mathbf{Z}) \cong \bigwedge^2 H^1(A^{an}, \mathbf{Z})$ defines a skew symmetric bilinear form

$$\psi_D : H_1(A^{an}, \mathbf{Z}) \times H_1(A^{an}, \mathbf{Z}) \to \mathbf{Z}$$

Here $L := H_1(A^{an}, \mathbf{Z})$ is a projective O_F-module $L \cong O_F \oplus \mathfrak{a}^{-1}$ of rank 2.

Now we impose some condition of symmetry for ψ_D. For $a \in O_F$, we consider the $\theta(a)$-multiplication $\theta(a) : A \to A$. Then for dual abelian variety A^* we have a functorially defined map $\theta(a)^* : A^* \to A^*$.

Definition 4.5 We say that $[D]$ is O_F-linear, if it satisfies the commutativity $\varphi_D \circ [a] = [a]^* \circ \varphi_D$, i.e., we have a commutative diagram

$$\begin{array}{ccc} A & \stackrel{\varphi_D}{\to} & A^* \\ \theta(a) \downarrow & & \downarrow \theta(a)^* \\ A & \stackrel{\varphi_D}{\to} & A^* \end{array}$$

This condition is equivalent to require that ψ_D is essentially O_F-linear or O_F-symmetric, i.e.,

$$\psi_D(\theta(a)\gamma_1, \gamma_2) = \psi_D(\gamma_1, \theta(a)\gamma_2)$$

for any $\gamma_1,\ \gamma_2 \in H^1(A^{an}, \mathbf{Z})$ and $a \in O_F$. Or equivalently $\psi_D : \bigwedge^2_{\mathbf{Z}} L \to \mathbf{Z}$ factors through the canonical quotient $\bigwedge^2_{\mathbf{Z}} L \to \bigwedge^2_{O_F} L \cong \mathfrak{a}^{-1}$. Therefore ψ_D is specified, if we choose a positive element $\beta \in \mathrm{Hom}(\mathfrak{a}^{-1}, \mathbf{Z}) \cong \mathfrak{a}\mathfrak{d}_F^{-1}$. Here $\mathfrak{d}_F^{-1}$ is the codifferent of F.

Thus for $A/\mathbf{C}$ which is O_F torus, a choice of polarization $\mathcal{C} = [D]$ is to choose a 'positive' element $\beta \in \mathfrak{a}\mathfrak{d}_F^{-1}$ corresponding to an ample divisor.

Remark For a generic A over $\mathbf{C}$ with O_F action, we have $\mathrm{NS}(A) \otimes_{\mathbf{Z}} \mathbf{Q} \cong F$. In this case, any polarization is automatically symmetric. But it may happen $F \subsetneq \mathrm{NS}(A) \otimes_{\mathbf{Z}} \mathbf{Q}$.

4.3 Weak polarization, or the notion of positivity

In the classical literature, the usual notion of polarization was to choose a $\mathbf{Q}_+$-ray in the ample cone $\mathrm{NS}^+(A)$. The notion of weak polarization is a variant of this. We want to consider O_F^+-ray.

Definition 4.6 Let $p : A \to S$ be an abelian scheme over a base scheme S with ring homomorphism $\theta : O_F \to \mathrm{End}_S(A)$. Then we say that A/S has O_F-multiplication, if the sheaf Lie(A/S) of the relative Lie algebra along the zero section $e : S \to A$ is a $O_F \otimes_{\mathbf{Z}} \mathcal{O}_S$-module of rank 1.

For simplicity, assume that the base $S = \mathrm{Spec}(k)$ of a field k. Then the Neron-Severi group $NS(A)$ is identified with the image of the map

$$D \mapsto \varphi_D \in \mathrm{Hom}(A, A^*).$$

Let $\mathrm{NS}(A)_{O_F}$ be the submodule of $\mathrm{NS}(A)$ consisting of O_F-linear classes. Then for a given class $[D] \in \mathrm{NS}(A)_{O_F}$, we can define an action of $a \in O_F$ by $\varphi_{a \cdot D} = \varphi_D \cdot \theta(a)$.

Definition 4.7 (Shimura, '63) Let E_F^+ be the group of totally positive units in F. Then set

$$\{ \bigcup_{\xi \in E_F^+} \xi \cdot [D] \}$$

is called "weak polarization", if $[D]$ is ample.

To justify this, one has to confirm that $\xi[D]$ is also ample if $[D]$ is ample and $\xi \in E_F^+$. This is almost obvious for $k = \mathbf{C}$ from the transcendental description of $[D]$ as a Riemann form.

Rapoport [10] call the equivalent notion as "positivity", which he attributes Deligne.

Now we want to define moduli functors to give 'Hilbert modular varieties.' But before that be have to confirm one point.

Proposition 4.3 *Let $p : A \to S$ be an abelian scheme with real multiplication θ over S. Then the etale sheaf $\mathcal{P}$, which associate to an etale cover $T \to S$ the set*

$$\mathrm{Hom}_{T,\ O_F}(A, A^*)^{sym} := \{\lambda : X_T \to X_T^* \mid \lambda : O_F\text{-linear and symmetric}\},$$

is locally constant with values in a projective O_F module of rank 1 equipped with the notion of 'positivity' corresponding to polarizations of (X, p).

Definition 4.8 *We denote by $F_{\mathfrak{a},\mathfrak{a}_+}$, a functor on $S \in (Schemes/\boldsymbol{Z})$ given by*

(i) : $p : A \to S$, an abelian scheme with real multiplication
$\theta : O_F \to \mathrm{End}_S(A)$,
(ii) rigidification on polarization data: $(\mathcal{P}, \mathcal{P}^+) \cong_\alpha (\mathfrak{a}^{-1}, \mathfrak{a}_+^{-1})$,
over S.

Theorem 4.1 (Rapoport [10], Theorem 1.20) *The functor $F_{\mathfrak{a},\mathfrak{a}_+}$ is representable by an algebraic stack $\mathcal{S}_{F_{\mathfrak{a},\mathfrak{a}_+}}$ smooth over $\boldsymbol{Z}$.*

Proof) The first step of the proof is to show the existence of the formal moduli space, i.e., to develop the local deformation theory for the moduli functor. Among others, the point is to show there are no obstructions of the deformation, which implies the formal etaleness and/or formal smoothness of the moduli problem in question.

Then 2-nd is to show algebraization of the formal moduli spaces, employing the approximation theory of M. Artin [1, 2, 3]. □

By the above theorem, we have models of Hilbert modular varieties defined over the rational number field $\mathbf{Q}$ by taking the associated coarse moduli space. But note here that the purpose of the paper [10] was to have toroidal compactifications such varieties.

We omit the discussion of the l-adic local systems and the automorphic line bundles over these moduli stacks. Some parts are just a formal analogue of the elliptic modular case.

The chapters IX and X of van der Geer [17] are regarded as very nice reviews before reading the original papers.

5 D: A problem

Here I want to suggest a problem. Let the base field $F = \mathbf{Q}(\sqrt{D})$ $(D > 0)$ be real quadratic, hence $g = 2$.

When the class number h_F^{strict} of the narrow sense of F is 1, i.e., the class number F is 1 and F has a unit with norm -1, the variety $GL_2(O_F)\backslash X_\infty$ is connected and naturally isomorphic to $SL_2(O_F)\backslash\mathbf{H}_{+,+} = GL_2^+(O_F)\backslash\mathbf{H}_{+,+}$, and F_{∞_i} $(i = 1, 2)$ are involutive automorphisms on it.

However when the class number of F is 1, but the norm of all units of F is 1, the class number in the narrow sense is 2. In this case, by Class Field Theory there is a quadratic extension K/F unramified at all the finite places, but ramified only at ∞_1, ∞_2, i.e., K is totally imaginary.

Then the variety $GL_2(O_F)\backslash X_\infty$ has 2 connected components:

$$S_{++} := GL_2^+(O_F)\backslash \mathbf{H}_{+,+}, \text{ and } S_{+-} := GL_2^+(O_F)\backslash \mathbf{H}_{+,-}.$$

Since either of F_{∞_i} induces a diffeomorphism between S_{++} and S_{+-}, these are homeomorphic complex analytic surfaces, but have different complex structure because the signature defects are different (Hirzebruch [16]). Probably, the surface S_{+-} has more holomorphic cusp forms than S_{++}. Here is a question.

Question What is the difference between S_{++} and S_{+-}? These two should be 'isospectral', probably have the same congruence zeta functions at good primes....

Remark When we obtain $S_{+,*}$ as a subvariety of a Siegel modular variety, $S_{+,-}$ is often obtained as a surface in the level 1 Siegel modular variety (*cf.* the classical examples by George Humbert). For S_{++} , we sometimes have to start from a para-modular varieties.

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E. Appendix 1: 代数群の整数論

過去のサマースクールの主題には、残念ながら「近似定理」や Reduction Theory など基本的な結果をやることはなかったようである。多変数保型形式のある程度一般的な結果に興味をもつものにとって、例えば代数群を主題とした集会のあとに、Reduction Theory を主題とする集まりがないのは極めて片手落ちに見える。Reduction Theory は現状の保型形式論では、まとまった形で一般的に完成してる唯一の大域理論である。もっとも、これは保型形式論というより、「代数群の整数論」といった方が適切かも知れない側面もあるが。これは、基本領域の体積の有限性、尖点の同値類の個数の有限性、保型形式の空間の次元の有限性など、極めて基本的な結果をもたらす。また Siegel 集合は、単に基本領域の体積の有限性のみならず、保型形式あるいは、その候補となる関数（あるいはヴェクトル束の切断）の基本的挙動を調べるには不可欠の道具である。実例を超えて、あるクラスの無限個の保型形式を問題にするときには不可欠である。もっとも、cusp 形式と、存在することが既に知られた正則な保型形式にのみ興味があるとなれば必要ないかも知れない。

ここでは、上記に基本的な結果に関心があり自習したい人のために文献案内をしたい。時間がないので、ここでは基本文献の簡単な紹介に留める。文献 [17]-[19] に詳細はある。

近似定理 (approximation theorem) は、このノートの文脈では次のようなことに用いた。B を体 F 上の quaternion algebra とする。$G = \mathrm{Res}_{F/\mathbf{Q}}(B^\times)$ を B の乗法群 $B^\times$ を表現する $\mathbf{Q}$ 代数群とする。即ち $\mathbf{Q}$ 上の algebra$R$ に対して、

$$R \mapsto (B \otimes_{\mathbf{Q}} R)^\times$$

という共変函手を考える。これを表現する group scheme が G である。

G のアデール化 $G(\mathbb{A})$ の有限部分 $G(\mathbb{A}_{fin})$ のコンパクト開部分群 K_f に対して、ある有限集合 $\{x_i \mid i \in I\}$ が存在して

$$G(\mathbb{A}) = \bigcup_{i \in I} G(\mathbf{Q})G(\mathbf{R})x_i K_f$$

という有限両側剰余類分解が存在することを主張している。群 G が一般線型群で、K_f が全ての素点で最大コンパクト開部分群であるときなどは、I の個数は考えている体の類数と一致して、I が「具体的」に求まる。いずれにせよ、これより、各 $i \in I$ に対して $\Gamma_i = G(\mathbf{Q}) \cap x_i\{G(\mathbf{R})K_f\}x_i^{-1}$ と置くとき、

$$G(\mathbf{Q})\backslash G(\mathbb{A})/K_f = \bigcup_{i \in I} \Gamma_i \backslash G(\mathbf{R})$$

となる。右辺が、通常考える連結な算術商である。

注意 問題をアデールの言葉で書くと、それだけで話が「大域化」したような虚しい幻想を抱く人がいるがそうではない。Γ の性質をきちんと反映している結果が大域的である。$\mathbf{C}$ 上算術商を最近は Shimura 多様体と呼ぶ人がいるが、昔は canonical model をこう呼んだ。$\mathbf{C}$ 上で考えるときには、連結成分が沢山あることに注意しないといけない。ナンセンスを書いている人がいる。特に幾何学的な状況とすり合わせるときに注意しないといけない。canonical model のときも、$\mathbf{Q}$ 上などで絶対既約ではないことに注意して下さい。

Borel [18] は前半は GL_n や SL_n を扱い「初等的に」かつ具体的に書いてある。後半は、ルート系など代数群の用語・道具を使う。Reduction Theory や Siegel 集合に親しめる。[19] は代数群の整数論の基本的な結果をまとめた本である。網羅的に書いてある。少し厳しい見方をすれば個性や特徴的な魅力がない (まあ教科書ですから)。それは充実した巻末の文献表のオリジナルに当たればよい。Chapter 4 が Reduction Theory で、Chapter 5 がアデール化を扱い、近似定理もここにある。強近似定理は Chapter 7 にある。Chapter 8 は類数である。

なお、この本も Borel の本も、不連続群は右側に書いてある。左派の私として不便である。

F. Appendix 2: 不連続群のコホモロジー

半単純 Lie 群の不連続群のコホモロジーに関しては、50 年代末から 60 年代前半に登場した、松島同型定理は衝撃的であったっと想像される。それ以前は、べき零群のときしか結果がなかったのだから。これは Eichler(-Shimura) 同型の高次元版で保型形式では極めて重要であるが、国内のサマースクールの講演対象になったことは、不幸にして大昔に織田が極めて準備のわるい話を２、３度したに留まる。幾何の人の勉強会にも出たが、60 年代始めまでの話しかない。

新しい話はなかったのか？そんなことはない。主として、70 年代の発展は、表現論の専門家がやっていた、相対 Lie 環コホモロジーの研究を除外すれば、Armand Borel が原動力になっていた。とくに cocompact でない不連続群のコホモロジーを扱うために Borel-Serre のコンパクト化、L^2 コホモロジー群など基本的な道具を作り出した。安定コホモロジー (stable cohomology) の理論は、代数体の K-群を計算するのに基本的な役割を果たした。しかし、K-理論の専門家の多くは (Ch. Soulé などは除くが) 証明を読んですらいないようである。

ここでは、大きな流れを二つ簡単に紹介する。

5.1 Automorphic cohomology and Eisenstein cohomology

$X_\infty = G/K$ を非コンパクト実半単純 Lie 群に付随するリーマン対称空間とする。算術的不連続部分群 Γ による商 $V_\Gamma = \Gamma\backslash X_\infty$ と G の有限次元有理表現 E を考える。このとき同型：

$$H^i(V_\Gamma,\ \tilde{E}) = H^i(\mathfrak{g}, K;\ C^\infty(\Gamma\backslash G) \otimes E)$$

の中で、右辺の $C^\infty(\Gamma\backslash G)$ を保型形式の空間 $\mathcal{A}(\Gamma\backslash G)$ で置き換える、即ち標準射

$$\mathcal{A}(\Gamma\backslash G) \to C^\infty(\Gamma\backslash G)$$

が quasi-isomorphism であることを期待するのは自然である。Borel も長い間努力をしていたが、最終的な解決は、Jens Franke [28] によって齎された。

なかなか難しい論文で、私はまだ良く理解していないので、興味あるかたはご自分でトライして織田に教えて下さい。

5.2 L^2-cohomology and Intersection cohomology

これは、80 年代に大きな関心を呼んだ話題である。E に自然に内積を導入する。X_∞ の不変計量と合わせて得られる、L^2 de Rham 複体によって、L^2 コホモロジー群 $H^i_{(2)}(V_\Gamma,\ \tilde{E})$ を考えるとき、同型：

$$H^i_{(2)}(V_\Gamma,\ \tilde{E}) \cong H^i(\mathfrak{g}, K; L^2(\Gamma\backslash G) \otimes E) \cong H^i(\mathfrak{g}, K; L^2(\Gamma\backslash G)_{dis} \otimes E)$$

が成立することは Borel が示した。これは、cocompact な Γ のよき世界を回復する簡単なやり方であるが、やはり写像：

$$H^i_c(V_\Gamma,\ \tilde{E}) \to H^i_{(2)}(V_\Gamma,\ \tilde{E}) \to H^i(V_\Gamma,\ \tilde{E})$$

を理解する最初の段階と見なすべきであろう。

他方、別の文脈から交叉コホモロジー (Intersection cohomology) の研究が、その少し後から展開した。Goresky-Macpherson [27] は特異点を持つ多様体に対して、いかに Poincaré 双対性定理を回復させるかという問題意識から、考えている多様体の特異点に付随する stratification に関してサイクルのサポートが適切に小さくなる条件 (perversity) を問題にし、これを使って、各種の perversity 1 をもつ鎖複体を定義し、相補的な perversity をもつコホモロジーに対して Poincaré 双対性を証明した。中間の perversity(middle perversity) が Poincaré 双対性を与える。後で Deligne が層の複体のコホモロジーによる再定義を与えてこの研究が一気に活性化した。Stephen Zucker は、対称領域の算術商の Satake, Baily-Borel コンパクト化 (最近は minimal compact 化という人もいる。それにしても人名をつけたり取ったりの基準は何だろう？) のときに、この交叉コホモロジーと、L^2 コホモロジーが自然に同型であることを予想した (1983 年頃)。これは、交叉コホモロジーの公理的な特徴づけを利用して、topological で比較的初等的な方針で、Saper-Stern と、斎藤盛彦の Hodge modules の理論を用いて Looijeng が、それぞれ独立に証明した。文献は E-Math で調べて下さい。このころこれは一部の人の間ですごく流行したが、それが終わると、まだ大きな問題が残っているが、何故か研究はその後下火になった。

この流行の時期に、いろいろ Shimura varietie の Hasse zeta 関数とか頻りに言及する人たちがいたので、一言注意しておく。

まず、Hasse zeta 関数は、織田の知る限り合同 zeta 関数の無限積であるので、先ず各素数に於ける合同 zeta 関数を理解する必要がある。通常は Shimura varieties の上のモジュラー形式の Hecke 多項式で書くことが目標になる。これは、Shimura varieties の reduction $mod\ \ p$ が良還元であるとして、コホモロジー群としては $H_c^*(V,\ \tilde{E})$ の etale cohomology 版と関係がある。ところが、

(1) これと $IH^*(V,\ \tilde{E})$ の差は、カスプから寄与する整数論的な情報を (単に組み合わせ論定でなく、境界成分の多様体の整数論的な情報を含むという意味で)、単に組み合わせ論的なものを超えた情報で記述されると思う根拠がある (これは極めて大切で深い問題である)、
という問題点と、

(2) 合同 zeta 関数を Hecke 多項式として理解できても、その無限積を大域的な Hasse として調べるのは、「解析」の問題である、
という問題がある。

何かことあるごとに Shimura varieties の Hasse zeta 関数とか言う人は、生命科学で、すぐ「この研究は、癌研究に役立つ」、とか「新薬開発に役立つ」とか言う人を何となく連想させる。

なお、L^2-コホモロジーの研究の濫觴は、松島-村上理論であると思う。Borel の論文には、たびたび Kuga の Lemma が使われる。

G: moduli 空間の構成の仕方

これは時代によって変遷している。Gauss が楕円モジュラー関数を考えていたのは、今日では知られている。楕円モジュラー関数を、楕円関数抜きで研究する、というのは Klein の提案らしい。A. Hurwitzs がどこかでそのように書いていたと思う。Fricke-Klein 本がその成果であろうか？ 同じころ Kiepert とか言う人が、楕円関数論を積極的に使って、楕円モジュラー関数を調べている。この人は Math. Annalen にたくさん論文を書いている。

20 世紀前半だと、Albert の Abel多様体の自己準同型環の研究とか (彼の論文には Riemann 行列 とか、Riemann 形式しかでてこない。つまり、アーベル多様体は間接的にしか姿を見せない) Lefschetz のサイクルの研究とか、Hodge の研究とかが、後のモジュライ空間の研究に役にたっている。代数曲線の場合だと、Teichmüller の研究とかもある。どちらかと言うと、解析的な手法で代数多様体も調べられた。余談であるが、Teichmüller が Nazi の SS であったとかで、「その理論をユダヤ系の人が多い 「New York 学派」(Ahlfors や Lipman Bers のなどのことを言うと思う) が発展させたのは、歴史の皮肉である」とは、New York のマンハッタン島で聞いた、故・久賀先生の評言である。

戦後 20 世紀の後半になると、代数多様体の純代数的な基礎理論ができて (Chevalley, Weil, Zariski, Samuel など？)、比較的初期に、Chow form, Chow coordinates (昔だと Samuel の教科書) など、moduli 空間を構成する道具が出来た。現在の functioriality による moduli の定義に相当する、specialization による井草の定義などがはっきり述べられたのはいつの頃か、失念しました。

1970 年頃までの、オリジナルの Shimura varieties の構成に使われた手法の概略は以下の通りである。そこで問題になったのは、最初はある自己準同型環を指定した、偏極アーベル多様体の族である。先ず、これを複素数体上で考えて、超越的な方法で、つまり周期行列 (の同値類) で (現在ならば Hodge 構造と言うが)、モジュライ空間を構成する。得られるものは、有界対称領域 $D = G/K$ の算術的不連続群 Γ による商 $V_\Gamma = \Gamma\backslash D$ である。これは、Satake, Bailey-Borel の

compact 化の理論より、複素数体 $\mathbf{C}$ 上の代数多様体になる。次に、この V_Γ (の関数体) に、$\mathbf{Q}$ 構造を入れる。これを Chow 座標を用いて行う。ここでは肝心の CM 点 (special points) による canonical model の特徴付けや、Deligne の命名では strange model (しかし創始者 Shimura とっては自慢に値すると明言していている)、PEL 型でない canonical model などには言及しない。ここでは、全体の手法が極めて体論的であることに注意して下さい。この後、登場するする手法ほど精密な結果を与えないが、目的によっては、最低のコストで必要な結果に到達できる利点が (現在でも) あります。

Moduli 空間の一般的な構成法は、A. Grothendieck の Hilbert schemes の導入によって一新した。これと Chow forms による Chow schemes の関連は適切なものを見て下さい。Hilbert schemes から moduli 空間を得るには、(相対) 不変式を用いる。David Mumford は, Geometric Invaraint Theory において、古典的な不変式論の復活、とくに商の存在に関わる基本的な問題 (Hilbert schemes の点で相対不変式が全て消えてしまうことの無いという)、stability の問題を深く追求した。後に Habousch が解決した、geometrically reductive algebraic groups の相対不変式環の有限生成と合わせて、この手法は極めて有力に見えた。事実、代数曲線、アーベル多様体の moduli 空間に関して、整数環 (を有限個の悪い素点で局所化したもの) の上で、moduli 空間を論じることができ、体論的な理論の almost all primes の限界を乗り越えることができた。(ベクトル束のモジュライ空間には、ここでは言及しない)。

しかし、考えている偏極代数多様体が定める、Hilbert schemes の点が、Geometric Invariant Theory 意味で stable, あるいは semistable, であることを判定することは、Gieseker の結果などがあるものの、今のところ純代数的な証明は難しい。有理数体など、標数 0 の体では、Shimura の手法の戻って、$\Gamma\backslash D$ の存在に依ることが出来るが、これでは整数環上の模型はつくれない。

そこで登場した対策が、「商」をつくる新しい方策 stack である。moduli 理論でも局所的に formal moduli の存在が言えないような、処理できない obstruction があるようなものは、元々現在の理論では扱えないので諦める。Formal moduli が存在すれば、moduli は局所的には formal には存在する。問題はそれを何か「代数的な」、有限性をもつものとして、scheme でなくとも、なにかもっと弱い意味で、schemes の「商」として構成しようと言うわけである。

これで、応用上充分な基礎が出来ているかどうかは織田は知らない。ただ何か対応する「解析的な結果」の支えなしでは、あまり実質的な応用は無いように思う。

以上の「歴史」と言うか「過程」と言うかは、多くの人にとって、前の世代の手法の忘却の歴史になっているようで、アルツハイマー症の人と話すように、相互に話しが通じなくなっている一因である。もっとも、本当の原因は、単にオリジナルの論文を読まないか、読めなくなったかが大きい。それはともかく、どのやり方にせよ、偏極の存在が基本で、それ故「射影幾何学」、つまり多様体の族を一度、射影多様体の族と考えていることに、注意して下さい。

蛇足を書きました。多変数保型形式、高次元モジュラー多様体を本当に研究なさろうという方に、少しでも役に立てば幸いです。

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