正標数楕円 $K3$ 曲面の数論と幾何について
On Arithmetic and Geometry of elliptic $K3$ surfaces in positive characeristic

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(Quasi-)Elliptic surfaces

$k = \bar{k}$: algebraically closed fields in characteristic $p > 0$

$f : X \longrightarrow C$: an elliptic surface over k

- X (resp. C) is a nonsingular projective surface (resp. curve) over k
- f is a surjective morphism whose general fiber is a nonsingular elliptic curve and fibers have no (-1) cueve (relatively minimal)
- f has a section O

$f : X \longrightarrow C$: a quasi-elliptic surface over k

- X (resp. C) is a nonsingular projective surface (resp. curve) over k
- f is a surjective morphism whose general fiber is a rational curve with a cusp
- f is relatively minimal and has a section O

Fundamental facts

Let $f : X \longrightarrow C$ be a (quasi-)elliptic surface

- Mordell-Weil group : $\mathrm{MW}(X/C) := \{\text{section of } f\}$
- Mordell-Weil theorem (Lang-Néron 1959, I. 1992)
  $\mathrm{MW}(X/C)$ is a finitely generated abelian group,
  i.e., $\mathrm{MW}(X/C) \cong \mathbb{Z}^{\oplus r} \oplus$ torsion subgroup
  For the case of unirational quasi-elliptic surfaces, $r = 0$ and torsion subgroup is p-elementary abelian group (I. 1992, 1994)
- $NS(X)$: Néron-Severi group of X (group of divisors on X up to algebraically equivalence)
- Relation between Mordell-Weil gruop $\mathrm{MW}(X/C)$ and Néron-Severi group (Shioda 1972, I. 1992) :

$$\mathrm{MW}(X/C) \cong NS(X)/T$$

  where T consists of O-sections, a general fiber, vertical divisors which donot intersect with O-section

Mordell-Weil lattices(Shioda 1990)

There exists unique group homomorphism

$$\varphi : \mathrm{MW}(X/C) \longrightarrow T^{\perp} \otimes \mathbb{Q} \subset NS(X) \otimes \mathbb{Q}$$

such that

1. for any $P \in \mathrm{MW}(X/C)$, $\varphi(P) \equiv (P) \mod T_{\mathbb{Q}}$,
2. $\mathrm{Ker}\,\varphi = \mathrm{MW}(X/C)_{tor}$,

then define a pairing $\langle P, Q \rangle$ for $P, Q \in \mathrm{MW}(X/C)$ by

$$\langle P, Q \rangle := -(\varphi(P), \varphi(Q)).$$

Then $(\mathrm{MW}(X/C)/(tor), \langle\ ,\ \rangle)$ is a positive definite lattice and we call it Mordell-Weil lattices (MWL) of X/C.

Mordell-Weil lattices (Shioda 1990)

The MWL $(\mathrm{MW}(X/C)/(tor), \langle\ ,\ \rangle)$ has a sublattice $(\mathrm{MW}(X/C)^\circ, \langle\ ,\ \rangle)$ which has a good properties such as even, integral and positive-definite.

- MWL for rational elliptic surface $X/\mathbb{P}^1$

$$
\begin{array}{ccccc}
\mathrm{MW}(X/\mathbb{P}^1)/(tor) & \cong & \overline{(T^\perp)^*} & \subset \langle O, F\rangle^\perp \cong E_8 \\
\cup & & \cup & \\
\mathrm{MW}(X/\mathbb{P}^1)^\circ & \cong & \overline{(T^\perp)} &
\end{array}
$$

$\Longrightarrow$ "Everything happens inside E_8 rational for elliptic surfaces"

Main subject

Motivation : When an object is related to $\cdots$ of order p, $\cdots$ of degree p, Frobenius morphism, $\cdots$, many interesting things happens in characteristic $p > 0$.

Want to study
elliptic $K3$ surfaces with p^n-torsion sections in characteristic $p > 0$.

- Want to classify them if possible.
- Want to study the geometry of their moduli.

Conclusion : Get the classification and interesting geometry.

P-torsion sections behave like irreducible divisors of the fibration. That is, the existence of p-torsion reduces the dimension of moduli.

Joint work with Christian Liedtke
"Elliptic $K3$ surface with p^n-torsion sections" (arXiv:1003.0144)

Related works on p-torsion sections

- (Levin '68) The order of torsion subgroup of non-constant elliptic cueve over a function field can be bouded by p and the genus of C.
- (Nguen-Saito '96, Hindry-Silberman '88, Goldfeld-Szpiro '95) the bound for prime-to-p part in terms of p and the gonality of $k(C)$
- (A. Schweizer '04) Case non-constant elliptic surface
  - the restriction on the genus of C, gonality of $k(C)$ and p-rank when $p^n > 11$,
  - the explicit examples including $K3$'s when $p^n \leq 11$.
- (Dolgachev-Keum '01) Using the theory of symplectic automorphism action on $K3$ in positve characteristic,
  - the order of symplectic auto. is prime to p when $p > 11$
  - the examples of $K3$'s which have symplectic auto. whose order is divisible by p when $p \leq 11$
- (Dolgache-Keum '09) If an elliptic $K3$ sufface has p-torsion section, then $p \leq 7$.

Main Theorem 1

Elliptic K3 surfaces with p^n-torsion section in characteristic p exist for $p^n \leq 8$ only.
If the fibration has constant j-invariant then $p^n = 2$.

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p^n	
$8 = 2^3$	
7	
5	
4	
3	
2	

Igusa moduli functor

Igusa moduli functor

$$[\mathrm{Ig}(p^n)^{\mathrm{ord}}] : (Sch/\mathbb{F}_p) \longrightarrow (Sets)$$

associates to every scheme S over $\mathbb{F}_p$ the pair of ordinary elliptic curve E over S and p^n-torsion section on the n-fold Frobenius pullback $(F^n)^*(E)$.

Theorem [Igusa '68]:

1. When $p^n \geq 3$ then $[\mathrm{Ig}(p^n)^{\mathrm{ord}}]$ is representable by a smooth affine curve ($=: \mathrm{Ig}(p^n)^{\mathrm{ord}}$), and we have the universal family $\mathcal{E} \longrightarrow \mathrm{Ig}(p^n)^{\mathrm{ord}}$.
2. The geometry of the normal compactification $\overline{\mathrm{Ig}(p^n)^{\mathrm{ord}}}$ of $\mathrm{Ig}(p^n)^{\mathrm{ord}}$ was studied.

The geometry of $\overline{\mathrm{Ig}(p^n)^{\mathrm{ord}}}$

$$\begin{tikzcd}
 & X = Y^{(p^n)} \arrow[rr] \arrow[dl] \arrow[dd] & & \bar{\mathcal{E}}^{(p^n)} \arrow[dl] \arrow[dd] & \\
Y \arrow[rr, "f"] \arrow[dd] & & \bar{\mathcal{E}} \arrow[dd] & & \\
 & \mathbb{P}^1 \arrow[rr, "\varphi"] \arrow[dl, "F^n"] & & \overline{\mathrm{Ig}(p^n)^{\mathrm{ord}}} \arrow[dl, "F^n"] & \\
\mathbb{P}^1 \arrow[rr, "\varphi"] & & \overline{\mathrm{Ig}(p^n)^{\mathrm{ord}}} \arrow[rr, "j"] & & \mathbb{P}^1
\end{tikzcd}$$

E.g., $n = 1$ and $p \geq 3$

- $j : \overline{\mathrm{Ig}(p^n)^{\mathrm{ord}}} \longrightarrow \mathbb{P}^1$ is Galois covering with $\mathbb{Z}/\frac{p-1}{2}\mathbb{Z}$
- j is totally ramified over the supersingular j-values and totally split over $j = \infty$,
  i.e., there are $(p-1)/2$ points (cusps) lying above ∞

The degenerating behavior of the universal family $\bar{\mathcal{E}}$ over the supersingular points and the cusps has been determined. (Liedtke-Schroeer '08)

Example : $\overline{\mathrm{Ig}(11)^{\mathrm{ord}}}$

- $\overline{\mathrm{Ig}(11)^{\mathrm{ord}}} \cong \mathbb{P}^1$
- $\overline{\mathrm{Ig}(11)^{\mathrm{ord}}}$ has 5 cusps
- our fibration has at least 5 fibres with multiplicative reduction
  $\Longrightarrow \bar{\mathcal{E}}$ has at least 5 fibres of type I_n, where 11 divides all these n's
- these contribute at least $5 \times (p-1) = 50$ to $\rho(X)$

Hence, not $K3$!.

Proof of Main Theorem 1

Case : constant j-invariant

- the p^n-torsion section is different from the zero section
  $\Longrightarrow$ the generic fiber is ordinary
  $\Longrightarrow$ the ordinary locus $U \subseteq \mathbb{P}^1$ is open and dense
- $p^n \geq 3 \Longrightarrow$ the Igusa moduli problem is representable
- constant j-invariant
  $\Longrightarrow$ the classifying morphism $\varphi : U \longrightarrow \mathrm{Ig}(p^n)^{\mathrm{ord}}$ is constant
- $X|U \longrightarrow U$ is a product family and not birational to a K3 surface

Hence in this case we have $p^n = 2$.

Proof of Main Theorem 1

Case : non-constant j-invariant

- the ordinary locus $U \subseteq \mathbb{P}^1$ is open and dense
- assume $p^n \geq 3$, i.e., that the Igusa moduli problem is representable
  $\Longrightarrow$ the classifying morphism $\varphi : U \longrightarrow \mathrm{Ig}(p^n)^{\mathrm{ord}}$ is dominant
  $\Longrightarrow$ $\mathrm{Ig}(p^n)^{\mathrm{ord}}$ is a rational curve
- $\mathrm{Ig}(p^n)^{\mathrm{ord}}$ is rational $\Longleftrightarrow$ $p^n \leq 11$ (Igusa '68)

Proof of Main Theorem 1

<u>Need to exclude the cases $p = 11$ and $p = 9$</u>

We have already excluded the case $p = 11$.

The remaining case $p^n = 9$ is excluded similarly.

- $\overline{\mathrm{Ig}(p^2)^{\mathrm{ord}}}$ has three cusps
  $\Longrightarrow$ our fibration has at least 3 fibres of type I_n, where p^2 divides all these n's
- these contribute at least $3 \times (p^2 - 1) = 24$ to $\rho(X)$,
  i.e., $b_2(X) \geq \rho(X) > 24$

Hence, X is not a K3 surface.

Elliptic $K3$ surfaces with p^n-torsion sections

p^n	$[\mathrm{Ig}(p^n)^{\mathrm{ord}}]$	$\deg \varphi$	description of the family
$8 = 2^3$	fine		
7	fine		
5	fine		
4	fine		
3	fine		
2	not fine		isotrivial case
			non-isotrivial case

Formal Brauer group and its height for $K3$ surfaces

- The functor on the category of finite local k-algebras A with residue field k

$$\widehat{\mathrm{Br}} \ : \ A \ \mapsto \ \ker\left(H^2_{\text{ét}}(X \times A, \mathbb{G}_m) \longrightarrow H^2_{\text{ét}}(X, \mathbb{G}_m)\right)$$

  is pro-represented by a smooth formal group of dim. $h^2(X, \mathcal{O}_X) = 1$, the *formal Brauer group* $\widehat{\mathrm{Br}}(X)$ of X. (Artin-Mazur '77)
- The height h of the formal Brauer group is ∞ or an integer $1 \leq h \leq 10$ and all values are taken. (Artin '74)
- Moreover, h determines the Newton polygon on second crystalline cohomology. (Illusie '79)
- In particular, the extreme cases are as follows:
  - $h = 1$ if and only if Newton- and Hodge- polygon coincide, i.e., the K3 surface is *ordinary*, and
  - $h = \infty$ if and only if the Newton polygon is a straight line, i.e., the K3 surface is *supersingular in the sense of Artin*.

The notion of supersingularity

Recall that a surface is called *supersingular in the sense of Shioda* if the rank of its Néron–Severi group is equal to its second Betti number.

- Unirational surfaces are supersingular in the sense of Shioda. (Shioda ’74)

On the other hand, a surface is called *supersingular in the sense of Artin* if its formal Brauer group has infinite height. (Artin ’74)

- Unirational K3 surfaces are supersingular in the sense of Artin.
- Supersingularity in the sense of Shioda implies supersingularity in the sense of Artin.

Some conjectures

For K3 surfaces,

- (Shioda) Shioda-supersingularity implies unirationality,
- (Artin) Artin-supersingularity implies unirationality,
- (Artin) Artin-supersingularity implies Shioda-supersingularity.

Remark: For elliptic K3 surfaces these two notions of supersingularity coincide (Artin '74).
In characteristic 2, there is another conjecture by Artin (Artin '74), which does not only imply the above conjectures but also gives a geometric explanation of the above conjectures:

- In characteristic 2, an elliptic fibration on a supersingular K3 surface arises via Frobenius pullback from a rational elliptic surface.

Remark : Such a conjecture cannot be true in general in characteristic $p \geq 3$.

Main Theorem 2 (Supersingular characterization)

Let $X \longrightarrow \mathbb{P}^1$ be an elliptic K3 surface with p-torsion sections in characteristic $p \geq 3$. Let $\varphi : \mathbb{P}^1 \longrightarrow \overline{\mathrm{Ig}(p)^{\mathrm{ord}}}$ be the compactified classifying morphism. Then the following are equivalent:

1. X arises as Frobenius pullback from a rational elliptic surface
2. X is unirational
3. X is supersingular
4. the fibration has precisely one fiber with additive reduction
5. φ is totally ramified over the supersingular point of $\overline{\mathrm{Ig}(p)^{\mathrm{ord}}}$

In particular, the conjectures of Artin and Shioda hold for this class of surfaces.

Remark : This theorem also valid for 4-torsion and 8-torsion sections.

Main Theorem 2' (Oridinary characterization)

Let $X \longrightarrow \mathbb{P}^1$ be an elliptic K3 surface with p-torsion sections in characteristic $p \geq 3$. Then the following are equivalent:

1. X arises as Frobenius pullback from a K3 surface
2. X is ordinary
3. X is not unirational
4. the fibration has precisely two fibers with additive reduction

Moreover, such surfaces can exist in characteristic $p \leq 5$ only.

Key Proposition A

Let $X \longrightarrow \mathbb{P}^1$ be an elliptic K3 surface with p-torsion section in characteristic p, whose fibration does not have constant j-invariant.

Then the fibration has at least oneand at most two fibres with potentially supersingular reduction. Moreover,

- if there is one fiber with potentially supersingular reduction then the formal Brauer group has height $h \geq 2$.
- if there are two fibers with potentially supersingular reduction then the formal Brauer group has height $h = 1$.

Proof:

- the fibration does not have constant j-invariant $\Longrightarrow$ the map from the base to the j-line is dominant, surj. $\Longrightarrow$ $\exists$ at least one fiber with pot. supersingular reduction
- Remaining assertions come from the followin Lemma on calculating the height of the formal Brauer group from the Weierstrass equation

$\square$

Lemma : Height of formal Brauer group

Let an elliptic $K3$ surface X be given by a Weierstraß equation

$$y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t)$$

where the $a_i(t)$'s are polynomials of degree $\leq 2i$,
i.e., $a_i(t) = \sum_{j=0}^{2i} a_{ij} t^j$.
Assume that $\underline{X \text{ have } p\text{-torsion sections}}$.
We can calculate the height h of the formal Brauer group as follows:

- For $p = 2$, $h = 1 \iff a_{11} \neq 0$
  $h \geq 2 \iff a_{11} = 0$
  $h \geq 3 \iff a_{11} = a_{33} = 0$
  $\cdots\cdots$ ([Artin 197]).
- For $p = 3$, $h = 1 \iff a_{11}^2 + a_{22} \neq 0$.
- For $p = 5$, $h = 1 \iff 2a_{44} \neq 0$.
- For $p = 7$: $\cdots\cdots$ by tedious calculation!

Proof of Lemma

$\underline{p = 2}$:

- potentially supersingular reduction over $t_0 \Leftrightarrow a_1(t_0) = 0$
- $K3 \Rightarrow \deg a_1(t) \leq 2 \Rightarrow$ at most two such fibers
- the fibration has two such fibres $\Rightarrow a_{11} \neq 0 \Rightarrow h = 1$
- the fibration has only one such fiber $\Rightarrow a_{11} = 0 \Rightarrow h \geq 2$

$\underline{p = 3}$:

- may assume $a_1(t) = 0$ after a suitable change of coordinates
- the Hasse invariant of the generic fiber is $-a_2(t) \in k(t)^{\times}/k(t)^{\times 2}$
- $\exists$ 3-torsion section $\Rightarrow$ the Hasse invariant is trivial, i.e., $-a_2(t)$ is a square
- fibers with potentially supersingular reduction fulfill $0 = a_2(t)^2$
- $\deg a_2(t) \leq 4 \Rightarrow$ there are at most two such fibers
- the fibration has two such fibers $\Leftrightarrow a_{22} \neq 0 \Leftrightarrow h = 1$

Proof of Lemma

$\underline{p = 5}$:

- may assume $a_1(t) = a_2(t) = a_3(t) = 0$
- $\exists$ 5-torsion sections $\Rightarrow 2a_4(t) =$ a fourth power (by computing the Hasse invariant)
- a fiber to have potentially supersingular reduction $\Rightarrow$ $2a_4(t) = 0$
- $\deg a_4(t) \leq 8$ and $2a_4(t) =$ a fourth power $\Rightarrow$ at most two such fibers
- $\exists$ two fibres with potentially supersingular reduction $\Leftrightarrow 2a_{44} \neq 0 \Leftrightarrow h = 1$.

by tedious calculation !

$\underline{p = 7}$: omit

$\square$

Key Proposition B

Let $X \longrightarrow B$ be an elliptic fibration with p-torsion sections and $p \geq 3$.

- Every additive fiber has potentially supersingular reduction.
- Every potentially supersingular fiber has additive reduction.

Very rought explanation of the proof :

First assertion (additive $\Longrightarrow$ potentially supersingular) is essentially by Liedtke-Schröer.

- Igusa moduli problem is fine when $p \geq 3$
- we know the degenerate properties of universal elliptic curves over $\mathrm{Ig}(p^n)^{\mathrm{ord}}$. (Liedtke-Schröer 08)
- look into j, and universal elliptic curve degenerates into multiplicative fibers at places of potentially multiplicative reduction
- only additive fibers can come from potentially supersingular places

Sketch of the proof of Key Proposition B

Second assertion (potentially supersingular $\Longrightarrow$ additive) is proved as follows:

- we know the degenerate properties of universal elliptic curves over $\mathrm{Ig}(p^n)^{\mathrm{ord}}$. (Liedtke-Schröer 08)
- calculating the intersectin pairing of O section and a p-torsion section
- translation by a p-torsion section give rise to a wild automorphism, and may apply the results of (Dolgachev-Keum '01) on symplectic automorphism action for $K3$ surfaces and its improved results on elliptic fibrations
- coclude that p-torsion sections are disjoint from O section when $p \geq 3$
- there does not exist good supersingular reduction

Main Theorem 2 (Supersingular characterization)

Let $X \longrightarrow \mathbb{P}^1$ be an elliptic K3 surface with p-torsion sections in characteristic $p \geq 3$. Let $\varphi : \mathbb{P}^1 \longrightarrow \overline{\mathrm{Ig}(p)^{\mathrm{ord}}}$ be the compactified classifying morphism. Then the following are equivalent:

1. X arises as Frobenius pullback from a rational elliptic surface
2. X is unirational
3. X is supersingular
4. the fibration has precisely one fiber with additive reduction
5. φ is totally ramified over the supersingular point of $\overline{\mathrm{Ig}(p)^{\mathrm{ord}}}$

In particular, the conjectures of Artin and Shioda hold for this class of surfaces.

Remark : This theorem also valid for 4-torsion and 8-torsion sections.

Proof of Main Theorem 2

- (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) holds in general.
- (3) $\Longrightarrow$ (4) comes from Key Propositions A and B.
- Equivalence between (4) and (5) is Key Proposition B.
- (5) $\Longrightarrow$ (1) :
  Since we know $X \longrightarrow \mathbb{P}^1$ arises as Frobenius pullback from some elliptic fibration $Y \longrightarrow \mathbb{P}^1$, we need to show that Y is rational.
  Let I_{pn_v}, $v = 1, \ldots$ be the multiplicative fibers. Since $p \geq 3$, the fibration does not have constant j-invariant and thus there exist places of potentially multiplicative reduction which are multiplicative. By Proposition B the potentially supersingular fiber is additive, say with m components and Swan conductor δ and we obtain

$$24 \ = \ c_2(X) \ = \ \sum_v pn_v \ + \ (2 + \delta + (m-1))$$

Proof of Main Theorem 2

We also know $X \longrightarrow \mathbb{P}^1$ arises as Frobenius pullback from some elliptic fibration $Y \longrightarrow \mathbb{P}^1$, which has multiplicative fibers I_{n_v}, $v = 1, \ldots$.
This fibration has one additive fiber also with Swan conductor δ and with, say, m' components. Thus we obtain

$$c_2(Y) = \sum_v n_v + (2 + \delta + (m' - 1)) \leq \frac{22 - \delta}{p} + (2 + \delta + (m' - 1))$$

Since $p \neq 2$, reduction of type I_n^* with $n \geq 1$ is potentially multiplicative and thus cannot occur as the additive fiber of $Y \longrightarrow \mathbb{P}^1$. Inspecting the list of additive fibers we obtain $m' \leq 9$.
On the other hand, Y is either rational or K3, i.e, $c_2(Y) = 12$ or $c_2(Y) = 24$.

Proof of Main Theorem 2

If $p \geq 5$ then $\delta = 0$ implies $c_2(Y) < 24$, which implies that Y is rational. If $p = 3$ then $c_2(Y) = 24$ could only be achieved if $\delta \geq 20$. However, since $\sum_n pn_v \geq p = 3$, this contradicts

$$24 = c_2(X) = \sum_v pn_v + (2 + \delta + (m - 1)).$$

Thus, Y is a rational surface also for $p = 3$. $\square$

Omit the proof of Theorem 2'.

Classifying morphism

For $p^n \geq 3$ and an elliptic $K3$ surface $f : X \longrightarrow \mathbb{P}^1$ with p^n-torsion sections, we have a diagram :

$$
\begin{tikzcd}
X = Y^{(p^n)} \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d] & \bar{\mathcal{E}} \arrow[d] & \\
\mathbb{P}^1 \arrow[r, "F^n"] & \mathbb{P}^1 \arrow[r, "\varphi"] & \overline{\mathrm{Ig}(p^n)^{\mathrm{ord}}} \arrow[r, "j"] & \mathbb{P}^1,
\end{tikzcd}
$$

where φ is the classifying morphism, j is the Galois covering induced by the j-invariant.

We are going to classify elliptic $K3$ surfaces with p^n-torsion sections using the classifying morphisim for each characteristic.

Especially, classify and study the geometry of supersingular elliptic $K3$ surfaces with p^n-torsion sections.

Classification in charctersitic 7

There exists only one elliptic K3 surface $X \longrightarrow \mathbb{P}^1$ with 7-torsion section in characteristic 7 up to isomorphism. It has the following invariants:

singular fibers	σ_0	$\mathrm{MW}^\circ(X)$	$\mathrm{MW}(X)$
$\mathrm{III}, 3 \times \mathrm{I}_7$	1	$A_1(7)$	$A_1^*(7) \oplus (\mathbb{Z}/7\mathbb{Z})$

The Weierstraß equation is given by the following:

$$y^2 = x^3 + tx + t^{12}.$$

In particular, it is the unique supersingular K3 surface with Artin invariant $\sigma_0 = 1$.

Classification in charctersitic 7

$\varphi : \mathbb{P}^1 \longrightarrow \overline{\mathrm{Ig}(7)^{\mathrm{ord}}}$: classifying morphism
$\bar{\mathcal{E}} \longrightarrow \overline{\mathrm{Ig}(7)^{\mathrm{ord}}}$: the universal curve

- $\deg \varphi \geq 2$ is impossible by an analysis of the multiplicative fibers
- hence φ is an isomorphism, proving uniqueness
- $\bar{\mathcal{E}}^{(7)}$ corresponds in fact a K3 surface, we get existence
- the singular fibres of $\bar{\mathcal{E}}/\mathrm{Ig}(7)^{\mathrm{ord}}$ are $(\mathrm{III}^*, 3 \times \mathrm{I}_1)$, trivial lattice $T_{\bar{\mathcal{E}}} = E_7$
- the singular fibres of $\bar{\mathcal{E}}^{(7)}/\mathrm{Ig}(7)^{\mathrm{ord}}$ are $(\mathrm{III}, 3 \times \mathrm{I}_7)$
- $\bar{\mathcal{E}}$ is rational, which implies that X is unirational

<u>How to determine the MWL of $\bar{\mathcal{E}}^{(7)}$</u> : The (full and narrow) Mordell-Weil lattices are $\mathrm{MW}(\bar{\mathcal{E}}) \cong A_1^*$ and $\mathrm{MW}^\circ(\bar{\mathcal{E}}) \cong A_1$ by the table. (Oguiso-Shioda '91)
Now, Frobenius induces an inclusion of lattices

$$\mathrm{MW}(\mathrm{Y})_{\mathrm{free}}(p) \subseteq \mathrm{MW}(\mathrm{X})_{\mathrm{free}},$$

which is of some finite index μ.

Classification in charctersitic 7

Taking determinants, we obtain

$$\mu^2 = \frac{\det \mathrm{MW}(Y)_{\mathrm{free}}(p)}{\det \mathrm{MW}(X)_{\mathrm{free}}}.$$

Since we have

$$\det \mathrm{NS}(X) = \frac{\det \mathrm{MW}(X)_{\mathrm{free}} \cdot \det T}{|\mathrm{MW}(X)_{\mathrm{tor}}|^2}$$

for elliptic surface whose j-invariant is not constant.

$$\mu^2 = \frac{\det A_1^*(7)}{\det \mathrm{NS}(X)\,|\mathrm{MW}(X)_{\mathrm{tor}}|^2} \det(U \oplus A_6^{\oplus 3} \oplus A_1) = \frac{\frac{1}{2}\cdot 7}{7^{2\sigma_0(X)}\cdot 7^2} \cdot 7^3 \cdot 2,$$

which yields $\mu = 1$. Thus, $\sigma_0 = 1$ and
$\mathrm{MW}(X) \cong A_1^*(7) \oplus (\mathbb{Z}/7\mathbb{Z})$. $\square$

Elliptic $K3$ surfaces with p^n-torsion sections

p^n	$[\mathrm{Ig}(p^n)^{\mathrm{ord}}]$	$\deg\varphi$	description of the family
$8 = 2^3$	fine		
7	fine	1	unique supersingular $K3$ ($\sigma_0 = 1$)
5	fine		
4	fine		
3	fine		
2	not fine		isotrivial case
			non-isotrivial case

Classification in charctersitic 5

Let $\varphi : \mathbb{P}^1 \longrightarrow \overline{\mathrm{Ig}(5)^{\mathrm{ord}}}$ be a classifying morphism. Then,

$$\deg \varphi = 2 \iff X \text{ is a K3 surface}$$

More precisely, the surfaces have the following invariants:

singular fibers	dim	σ_0	$\mathrm{MW}^\circ(X)$	$\mathrm{MW}(X)$
$2 \times \mathrm{II}, 4 \times \mathrm{I}_5$	2			
$2 \times \mathrm{II}, \mathrm{I}_{10}, 2 \times \mathrm{I}_5$	1			
$2 \times \mathrm{II}, 2 \times \mathrm{I}_{10}$	0			
$\mathrm{IV}, 4 \times \mathrm{I}_5$	1	2	$A_2(5)$	$A_2^*(5) \oplus \mathbb{Z}/5\mathbb{Z}$
$\mathrm{IV}, \mathrm{I}_{10}, 2 \times \mathrm{I}_5$	0	1	$\langle 30 \rangle$	$\langle \frac{5}{6} \rangle \oplus \mathbb{Z}/5\mathbb{Z}$

Here, dim denotes the dimension of the family. For the supersingular surfaces, this list also gives Artin invariants σ_0 and their (narrow) Mordell–Weil lattices.

Classification in charctersitic 5

Proof: (Similar to the characteristic 7 case.)

- The universal elliptic curve over the Igusa curve $\bar{\mathcal{E}} \longrightarrow \overline{\mathrm{Ig}(5)^{\mathrm{ord}}}$ is given by the Weierstraß

$$y^2 = x^3 + 3t^4x + t^5,$$

  which has a singular fiber of type II^* over $t = 0$ and fibers of type I_1 over $t = \pm 1$.
- Note that this surface is a rational extremal elliptic surface.
- We write the classifying morphism $\varphi = \varphi_{\alpha\beta} : \overline{\mathrm{Ig}(5)^{\mathrm{ord}}} \longrightarrow \mathbb{P}^1$ as

$$t = \frac{\alpha s^2 + \beta}{s^2 + 1}$$

  whose branch points are $t = \alpha$ and $t = \beta$, where t (resp. s) is a local parameter of $\mathbb{P}^1$ (resp. $\overline{\mathrm{Ig}(5)^{\mathrm{ord}}}$).

Classification in charctersitic 5

- Then our surfaces arise as pullbacks along F and $\varphi_{\alpha\beta}$:

$$\begin{tikzcd} X = Y^{(p)} \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathbb{P}^1 \arrow[r, "F"] & \mathbb{P}^1 \arrow[r, "\varphi_{\alpha\beta}"] & \overline{\mathrm{Ig}(5)^{\mathrm{ord}}} \end{tikzcd}$$

- The elliptic surface Y is given by the Weierstraß equation

$$y^2 \;=\; x^3 \;+\; 3(\alpha s^2 + \beta)^4 x \;+\; (\alpha s^2 + \beta)^5\,(s^2 + 1),$$

and depending on α and β we obtain the following list giving the explicit classification of our surfaces.

$\{\alpha,\beta\} \cap \{0, \pm 1\}$	singular fibers of X	singular fibers of Y	Y
$\emptyset$	$2 \times \mathrm{II}, 4 \times \mathrm{I}_5$	$2 \times \mathrm{II}^*, 4 \times \mathrm{I}_1$	K3
$\{1\}, \{-1\}$	$2 \times \mathrm{II}, \mathrm{I}_{10}, 2 \times \mathrm{I}_5$	$2 \times \mathrm{II}^*, \mathrm{I}_2, 2 \times \mathrm{I}_1$	K3
$\{1, -1\}$	$2 \times \mathrm{II}, 2 \times \mathrm{I}_{10}$	$2 \times \mathrm{II}^*, 2 \times \mathrm{I}_2$	K3
$\{0\}$	$\mathrm{IV}, 4 \times \mathrm{I}_5$	$\mathrm{IV}^*, 4 \times \mathrm{I}_1$	rational
$\{0, 1\}, \{0, -1\}$	$\mathrm{IV}, \mathrm{I}_{10}, 2 \times \mathrm{I}_5$	$\mathrm{IV}^*, \mathrm{I}_2, 2 \times \mathrm{I}_1$	rational

Classification in charctersitic 5

By Main Theorem 2 the supersingular surfaces are precisely those that arise as Frobenius pullbacks from rational elliptic surfaces. It remains to determine the Mordell–Weil groups and Artin invariants. On can use the similar argument as characteristic 7. (Omit it.) $\square$

Remark : Since Y has two singular fibers of type II^* when it is $K3$ surface, we can apply Shioda's sandwich theorem for studying it.

Further question : Is the supersingular family complete ?

Proposition for completeness

Let X be an elliptic K3 surface with p^n-torsion section in characteristic p. Assume that X is supersingular with Artin-invariant σ_0.
Then, every K3 surface which is supersingular in the sense of Shioda with Artin invariant σ_0 in characteristic p possesses an elliptic fibration with p^n-torsion section.

Proof:

- a (quasi-)elliptic fibration on $X \Longleftrightarrow U \hookrightarrow \mathrm{NS}(X)$ (isometry)
  (U : a hyperbolic lattice of rank 2)
- the trivial lattice T is the sub-lattice of $\mathrm{NS}(X)$ generated by U and all $x \in U^{\perp}$ with $x^2 = -2$
- the torsion sections of the fibration correspond to the torsion of $\mathrm{NS}(X)/T$

Proof of Proposition for completeness

- the Néron–Severi group of a (Shioda-)supersingular K3 surface is uniquely determined by p and σ_0 (Rudakov Shafarevich 1979)
- one of these surfaces possesses a (quasi-)elliptic fibration with p^n-torsion section $\Rightarrow$ so do all of them
- need to check the genus 1 fibration on another K3 surface Y with the same p and σ_0 corresponding to $U \hookrightarrow \mathrm{NS}(X)$ is elliptic, not quas-elliptic.
- if $p \geq 5$ or if $\mathrm{rank}(T) < 22$ then the fibration on Y is automatically elliptic and the quasi-elliptic case cannot occur at all
- if $p \leq 3$ and $\mathrm{rank}(T) = 22$ then the elliptic fibration on X is extremal and these K3 surfaces have been explicitly classified (I. 2002)
- these surfaces have Artin invariant $\sigma_0 = 1$, i.e., X is isomorphic to Y

$\square$

Completeness results in characteristic 5

Every (Shioda-)supersingular K3 surface with $\sigma_0 \leq 2$ in characteristic 5 possesses an elliptic fibration with 5-torsion section.

Elliptic $K3$ surfaces with p^n-torsion sections

p^n	$[\mathrm{Ig}(p^n)^{\mathrm{ord}}]$	$\deg \varphi$	description of the family
$8 = 2^3$	fine		
7	fine	1	unique supersingular $K3$ ($\sigma_0 = 1$)
5	fine	2	2-dim ordinary $K3$'s $\supset$ 1-dim s.s. $K3$'s ($\sigma_0 \leq 2$)
4	fine		
3	fine		
2	not fine		isotrivial case
			non-isotrivial case

Classification in charctersitic 3

Let $\varphi : \mathbb{P}^1 \longrightarrow \overline{\mathrm{Ig}(3)^{\mathrm{ord}}}$ be a classifying morphism. Then,

$$2 \leq \deg \varphi \leq 6 \Longleftrightarrow X \text{ is a K3 surface}$$

More precisely,

1. $\deg \varphi = 2$ and $\varphi^{-1}(O)$ consists of two points.
2. $\deg \varphi = 3$, φ is separable and $\varphi^{-1}(O)$ consists of two points.
3. $\deg \varphi = 4$ and $\varphi^{-1}(O)$ consists of one or two points.
4. $\deg \varphi = 5$ and $\varphi^{-1}(O)$ consists of one point or two points with ramification index $e = 2$ and $e = 3$.
5. $\deg \varphi = 6$ and $\varphi^{-1}(O)$ consists of one point or two points with ramification index $e = 3$.

Conversely, if φ is as above then the associated elliptic fibration with 3-torsion section is a K3 surface.
We denote by $O \in \overline{\mathrm{Ig}(3)^{\mathrm{ord}}}$ the unique supersingular point.

Classification in charctersitic 3

Every (Shioda-)supersingular K3 surface with Artin invariant $\sigma_0 \leq 6$ in characteristic 3 possesses an elliptic fibration with 3-torsion section.
The complete list of these (supersingular) surfaces is given by the following table:

Supersingular $K3$ surfaces in charctersitic 3

$\deg \varphi = 6$ (separable)

singular fibers	dim	σ_0	$\mathrm{MW}^\circ(X)$	$\mathrm{MW}(X)$
$\mathrm{II}_4, 6 \times \mathrm{I}_3$	5	6	$E_8(3)$	$E_8(3) \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{II}_4, \mathrm{I}_6, \mathrm{I}_3 \times 4$	4	5	$E_7(3)$	$E_7^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{II}_4, \mathrm{I}_9, \mathrm{I}_3 \times 3$	3	4	$E_6(3)$	$E_6^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{II}_4, \mathrm{I}_6 \times 2, \mathrm{I}_3 \times 2$	3	4	$D_6(3)$	$D_6^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{II}_4, \mathrm{I}_{12}, \mathrm{I}_3 \times 2$	2	3	$D_5(3)$	$D_5^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{II}_4, \mathrm{I}_6 \times 3$	2	3	$D_4(3) \oplus A_1(3)$	$D_4^*(3) \oplus A_1^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{II}_4, \mathrm{I}_9, \mathrm{I}_6, \mathrm{I}_3$	2	3	$A_5(3)$	$A_5^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{II}_4, \mathrm{I}_{15}, \mathrm{I}_3$	1	2	$A_4(3)$	$A_4^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{II}_4, \mathrm{I}_{12}, \mathrm{I}_6$	1	2	$A_3(3) \oplus A_1(3)$	$A_3^*(3) \oplus A_1^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{IV}_2, 6 \times \mathrm{I}_3$	4	5	$E_6(3)$	$E_6^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{IV}_2, \mathrm{I}_6, \mathrm{I}_3 \times 4$	3	4	$A_5(3)$	$A_5^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{IV}_2, \mathrm{I}_9, \mathrm{I}_3 \times 3$	2	3	$A_2(3)^{\oplus 2}$	$A_2^*(3)^{\oplus 2} \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{IV}_2, \mathrm{I}_6 \times 2, \mathrm{I}_3 \times 2$	2	3	$L_4(3)$	$L_4^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{IV}_2, \mathrm{I}_{12}, \mathrm{I}_3 \times 2$	1	2	$L_3(3)$	$L_3^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{IV}_2, \mathrm{I}_6 \times 3$	1	2	$A_1(3) \oplus L_2(3)$	$A_1^*(3) \oplus L_2^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{I}_{0,0}^*, \mathrm{I}_3 \times 6$	3	4	$D_4(3)$	$D_4^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{I}_{0,0}^*, \mathrm{I}_6, 4 \times \mathrm{I}_3$	2	3	$A_1(3)^{\oplus 3}$	$A_1^*(3)^{\oplus 3} \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{I}_{0,0}^*, \mathrm{I}_6 \times 2, \mathrm{I}_3 \times 2$	1	2	$A_1(3)^{\oplus 2}$	$A_1^*(3)^{\oplus 2} \oplus \mathbb{Z}/6\mathbb{Z}$
$\mathrm{I}_{0,0}^*, \mathrm{I}_9, \mathrm{I}_3 \times 3$	1	2	$L_2(3)$	$L_2^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{I}_{0,0}^*, \mathrm{I}_6 \times 3$	0	1	$A_1(3)$	$A_1^*(3) \oplus \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
$\mathrm{I}_{0,0}^*, \mathrm{I}_{12}, \mathrm{I}_3 \times 2$	0	1	$\langle 12 \rangle$	$\langle \frac{3}{4} \rangle \oplus \mathbb{Z}/6\mathbb{Z}$

Supersingular $K3$ surfaces in charctersitic 3

$\deg \varphi = 6$ (inseparable)

singular fibers	dim	σ_0	$\mathrm{MW}^\circ(X)$	$\mathrm{MW}(X)$
$\mathrm{IV}_2, 2 \times \mathrm{I}_9$	1	2	$A_2(3)$	$A_2^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{IV}_2, \mathrm{I}_{18}$	0	1	$\langle 18 \rangle$	$\langle \frac{1}{2} \rangle \oplus \mathbb{Z}/3\mathbb{Z}$

$\deg \varphi = 5$

singular fibers	dim	σ_0	$\mathrm{MW}^\circ(X)$	$\mathrm{MW}(X)$
$\mathrm{IV}_5, 5 \times \mathrm{I}_3$	4	5	$E_8(3)$	$3.(E_8(3)) \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{IV}_5, \mathrm{I}_6, 3 \times \mathrm{I}_3$	3	4	$E_7(3)$	$3.(E_7^*(3)) \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{IV}_5, 2 \times \mathrm{I}_6, \mathrm{I}_3$	2	3	$D_6(3)$	$3.(D_6^*(3)) \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{IV}_5, \mathrm{I}_9, 2 \times \mathrm{I}_3$	2	3	$E_6(3)$	$3.(E_6^*(3)) \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{IV}_5, \mathrm{I}_9, \mathrm{I}_6$	1	2	$A_5(3)$	$3.(A_5^*(3)) \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{IV}_5, \mathrm{I}_{12}, \mathrm{I}_3$	1	2	$D_5(3)$	$3.(D_5^*(3)) \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{IV}_5, \mathrm{I}_{15}$	0	1	$A_4(3)$	$3.(A_4^*(3)) \oplus \mathbb{Z}/3\mathbb{Z}$

Supersingular $K3$ surfaces in charctersitic 3

$\deg \varphi = 4$

singular fibers	dim	σ_0	$\mathrm{MW}^{\circ}(X)$	$\mathrm{MW}(X)$
$\mathrm{IV}_4^*, 4 \times \mathrm{I}_3$	3	4	$E_6(3)$	$E_6^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{IV}_4^*, \mathrm{I}_6, 2 \times \mathrm{I}_3$	2	3	$A_5(3)$	$A_5^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{IV}_4^*, \mathrm{I}_6, \mathrm{I}_6$	1	2	$L_4(3)$	$L_4^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{IV}_4^*, \mathrm{I}_9, \mathrm{I}_3$	1	2	$A_2(3)^{\oplus 2}$	$A_2^*(3)^{\oplus 2} \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathrm{IV}_4^*, \mathrm{I}_{12}$	0	1	$L_3(3)$	$L_3^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$

Here, L_2, L_3, and L_4 are lattices of rank 2, 3, 4, all of determinant 12, whose matrices are given by

$$L_2 = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}, L_3 = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 4 \end{pmatrix}, L_4 = \begin{pmatrix} 4 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 1 & 0 & -1 & 2 \end{pmatrix}.$$

Also, the notation $3.L$ for a lattice L stands for a lattice that has L as a sublattice of index 3.

Proof and related geometry

Let us concentrate on supersingular $K3$ surfaces with 3-torsion sections.

- the classifying morphism φ is totally ramified over the supersingular point $O \in \overline{\mathrm{Ig}(3)^{\mathrm{ord}}}$
- this gives $4 \leq \deg \varphi \leq 6$

<u>Explicit equations of Y</u> :
let $f_3(s)$, $f_4(s)$ and $f_5(s)$ be polynomials of degree 3, 4 and 5 with no zero in $s = 0$. Then we substitute

$$t = \frac{s^6}{f_5(s)}, \; t = \frac{s^5}{f_4(s)} \quad \text{and} \quad t = \frac{s^4}{f_3(s)}$$

into the Weierstraß equation $y^2 + txy = x^3 - t^5$ of the universal family over $\overline{\mathrm{Ig}(3)^{\mathrm{ord}}}$.
In all cases this leads to a Weierstraß equation

$$y^2 = x^3 + s^2x^2 + s^5 + r_4s^4 + r_3s^3 + r_2s^2 + r_1s + r_0$$

for certain $(r_4, r_3, r_2, r_1, r_0) \in \mathbb{A}_k^5$.

Proof and related geometry

Depending on the degree of φ these coefficients satisfy the following conditions:

$$\begin{array}{lcl} \deg\varphi = 6 & : & r_1 r_0 \neq 0 \\ \deg\varphi = 5 & : & r_1 \neq 0, \quad r_0 = 0 \\ \deg\varphi = 4 & : & r_2 \neq 0, \quad r_1 = r_0 = 0 \end{array}$$

<u>Relation with the semi-universal deformation of a RDP</u>:
It is remarkable that these rational elliptic surfaces appear in the family of elliptic surfaces related to the semi-universal deformation of the E_8^2-singularity in characteristic 3, which is given by

$$y^2 = x^3 + (t^2+s)x^2 + (q_1 t + q_0)x + t^5 + r_4 t^4 + r_3 t^3 + r_2 t^2 + r_1 t + r_0.$$

To obtain elliptic K3 surfaces with 3-torsion section we have to take the Frobenius pullback of these surfaces.

Proof and related geometry

Then the non-trivial 3-torsion sections of the fibration are explicitly given by

$$(-(\tau^5 + r_4^{\frac{1}{3}}\tau^4 + r_3^{\frac{1}{3}}\tau^3 + r_2^{\frac{1}{3}}\tau^2 + r_1^{\frac{1}{3}}\tau + r_0^{\frac{1}{3}}),$$

$$\pm\tau^3(\tau^5 + r_4^{\frac{1}{3}}\tau^4 + r_3^{\frac{1}{3}}\tau^3 + r_2^{\frac{1}{3}}\tau^2 + r_1^{\frac{1}{3}}\tau + r_0^{\frac{1}{3}}))$$

(Needs to modify slightly for $\deg\varphi = 4$.)

Calculation of the MWL's and the Artin invariants:

- the index of $\mathrm{MW}(Y)_{\mathrm{free}}(3)$ inside $\mathrm{MW}(X)_{\mathrm{free}}$ is related to the Artin invariant of X for each case in the table
  $\Longrightarrow$ obtain an upper bound for the Artin invariant
- all the surfaces in the table can be realized inside the family corresponding to the semi-universal deformation of the E_8^2-singularity
  $\Longrightarrow$ the dimension of the surface having the given type of singular fibers inside the moduli space is bounded from below

Proof and related geometry

- gives the Artin invariants for the cases $\deg\varphi = 4$ and $\deg\varphi = 6$.
- need a more precise analysis for the case $\deg\varphi = 5$
  - X : an elliptic K3 surface with 3-torsion sections whose singular fibers are of type $\mathrm{IV}_5, 5 \times \mathrm{I}_3$
    $\Longrightarrow \mu^2 = 3^{12-2\sigma_0(X)}, \mu = [\mathrm{MW}(X)_{\mathrm{free}} : \mathrm{MW}(Y)_{\mathrm{free}}(3)]$
    $\Longrightarrow \sigma_0(X) \leq 6$.
  - these surfaces are realized inside the semi-universal deformation of the E_8^2-singularity
    $\Longrightarrow \sigma_0(X) \geq 5$
  - have to decide whether $\mu = 1$ or $\mu = 3$ holds true
  - assume $\mu = 1$
  - $\mathrm{MW}(Y)_{\mathrm{free}} = \mathrm{MW}^{\circ}(Y) = E_8 \Longrightarrow \mathrm{MW}(X)_{\mathrm{free}} = \mathrm{MW}^{\circ}(X) = E_8(3)$
  - the 3-torsion sections of this surface do not lie in $\mathrm{MW}^{\circ}(X) \Longrightarrow$ a contradiction
  - $\mu = 3$ and $\sigma_0(X) = 5$
- every (Shioda-)supersingular K3 surface with $\sigma_0 \leq 6$ possesses an elliptic fibration with 3-torsion section

Elliptic $K3$ surfaces with p^n-torsion sections

p^n	$[\mathrm{Ig}(p^n)^{\mathrm{ord}}]$	$\deg\varphi$	description of the family
$8 = 2^3$	fine		
7	fine	1	unique supersingular $K3$ ($\sigma_0 = 1$)
5	fine	2	2-dim ordinary $K3$'s
			$\supset$ 1-dim s.s. $K3$'s ($\sigma_0 \leq 2$)
4	fine		
3	fine	$2, \cdots, 6$	6-dim ordinary $K3$'s
			$\supset$ 5-dim s.s. $K3$'s ($\sigma_0 \leq 6$)
2	not fine		isotrivial case
			non-isotrivial case

Classification in charctersitic 2, 2-torsion sections

Case: constant j-invariant

1. one additive fiber of type $\mathrm{I}^*_{12,6}$, and then $h \geq 2$, or
2. two additive fibers, both of type $\mathrm{I}^*_{4,2}$, and then $h = 1$.

Case : non-constant j-invariant

1. the fibration has precisely one additive fiber, which is potentially supersingular. In this case $h \geq 2$ holds true.
2. the fibration is semi-stable and there is precisely one fiber with good and supersingular reduction. Moreover, X is unirational and $h = \infty$.
3. the fibration has precisely two fibers with additive reduction, both of which are potentially supersingular. In this case $h = 1$ holds true.
4. the fibration has precisely two fibers with additive reduction, one of which is potentially supersingular and the other one is potentially ordinary of type $\mathrm{I}^*_{4,2}$. In this case $h = 1$ holds true.

Classification in charctersitic 2, 2-torsion sections

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4. the fibration has precisely two fibers with additive reduction, one of which is potentially supersingular and the other one is potentially ordinary of type $\mathrm{I}^*_{4,2}$. In this case $h = 1$ holds true.

Theorem for constant j-invariant case

Every elliptically fibered K3 surface with constant j-invariant and 2-torsion section in characteristic 2 arises as minimal desingularization of

$$(E_1 \times E_2)/G \longrightarrow E_2/G \cong \mathbb{P}^1, \tag{1}$$

where E_1 is an ordinary and E_2 is an arbitrary elliptic curve, and $G \cong \mathbb{Z}/2\mathbb{Z}$ acts via the sign involution on each factor.
Conversely, for any two elliptic curves E_1, E_2, where E_1 is ordinary, a minimal desingularization of (1) yields an elliptic K3 surface with constant j-invariant and 2-torsion section. More precisely,

E_2	singular fibers	ρ	h
ordinary	$2 \times \mathrm{I}_{4,2}^*$	$18 \leq \rho \leq 20$	1
supersingular	$\mathrm{I}_{12,6}^*$	18	2

In particular, these surfaces cannot be supersingular, and $h = 2$ is possible.

Classification in charctersitic 2, 2-torsion sections

Case: constant j-invariant

1. one additive fiber of type $\mathrm{I}^*_{12,6}$, and then $h \geq 2$, or
2. two additive fibers, both of type $\mathrm{I}^*_{4,2}$, and then $h = 1$.

Case : non-constant j-invariant

1. the fibration has precisely one additive fiber, which is potentially supersingular. In this case $h \geq 2$ holds true.
2. the fibration is semi-stable and there is precisely one fiber with good and supersingular reduction. Moreover, X is unirational and $h = \infty$.
3. the fibration has precisely two fibers with additive reduction, both of which are potentially supersingular. In this case $h = 1$ holds true.
4. the fibration has precisely two fibers with additive reduction, one of which is potentially supersingular and the other one is potentially ordinary of type $\mathrm{I}^*_{4,2}$. In this case $h = 1$ holds true.

E_8^4-family

Elliptic $K3$ surfaces with 2-torsion sections in the cases 1 and 2 in non-constant j-invariant are realized using E_8^4-family.
Consider the singularity defined by the affine equation in $\mathbb{A}_k^3$:

$$E_8^4 : y^2 + txy = x^3 + t^5$$

and the semiuniversal deformation of this singularity with parameter $\lambda = (p_0, p_1, q, r_4, r_3, r_2, r_1, r_0) \in \mathbb{A}_k^8$,

$$y^2 + txy + (p_0 + p_1 t)y = x^3 + qx + t^5 + r_4 t^4 + r_3 t^3 + r_2 t^2 + r_1 t + r_0.$$

Taking Kodaira-Néron model,
we get the elliptic surface $f_\lambda : X_\lambda \longrightarrow \mathbb{P}^1$.
Inside the parameter space $\mathbb{A}_k^8$, let us write the semistable locus as $\mathcal{S}$,

$$\mathcal{S} := \{\lambda \in \mathbb{A}_k^8 | p_0 \neq 0\}.$$

E_8^4-family

We pick up special members which dominate other sub-members from E_8^4-family.

For $\lambda \in \mathcal{S}$, an elliptic surface X_λ is called a *basic member* if the singular fibers consist of one I_n with some $1 \leq n \leq 9$ and $(12 - n)$ I_1's. We write this X_λ as $X(\mathrm{I}_n)$.

For $\lambda \in \mathbb{A}_k^8 - \mathcal{S}$, an elliptic surface X_λ is called a *basic member* if the singular fiber over $t = 0$ is of additive type T and all the other singular fibers are all I_1's. We write this X_λ as $X(T)$.

Semistable locus $\mathcal{S}$ of $\mathbb{A}_k^8$ has the stratification as follows:

$$\mathbb{A}_k^8 \supset \mathcal{S} = \mathcal{X}(\mathrm{I}_1) \supset \mathcal{X}(\mathrm{I}_2) \supset \cdots \supset \mathcal{X}(\mathrm{I}_8) \supset \mathcal{X}(\mathrm{I}_9)$$

Each stratum $\mathcal{X}(\mathrm{I}_l)(l > 1)$ has codimension 1 inside $\mathcal{X}(\mathrm{I}_{l-1})$, thus a stratum $\mathcal{X}(\mathrm{I}_l)$ has dimension $9 - l$.

E_8^4-family

A general member of each stratum has the following Mordell-Weil lattices:

Type of singular fiber	MWL	narrow MWL
$\mathrm{I}_1 \times 12$	E_8	E_8
$\mathrm{I}_2, \mathrm{I}_1 \times 10$	E_7^*	E_7
$\mathrm{I}_3, \mathrm{I}_1 \times 9$	E_6^*	E_6
$\mathrm{I}_4, \mathrm{I}_1 \times 8$	D_5^*	D_5
$\mathrm{I}_5, \mathrm{I}_1 \times 7$	A_4^*	A_4
$\mathrm{I}_6, \mathrm{I}_1 \times 6$	$A_2^* \oplus A_1^*$	$A_2 \oplus A_1$
$\mathrm{I}_7, \mathrm{I}_1 \times 5$	$\frac{1}{7}\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$	$\begin{pmatrix} 4 & -2 \\ -1 & 2 \end{pmatrix}$
$\mathrm{I}_8, \mathrm{I}_1 \times 4$	$\langle \frac{1}{8} \rangle$	$\langle 8 \rangle$
$\mathrm{I}_9, \mathrm{I}_1 \times 3$	$\mathbb{Z}/3\mathbb{Z}$	$\{0\}$

Frobenius base change gives the case 1

For a general member of $\mathcal{X}(\mathrm{I}_l)$ with $1 \le l \le 9$, we take Frobenius base change once to get the new family $\widetilde{\mathcal{X}(\mathrm{I}_l)}$. A general member $\widetilde{X(\mathrm{I}_l)}$ of $\widetilde{\mathcal{X}(\mathrm{I}_l)}$ is a supersingular elliptic $K3$ surface with the following data:

l	Type of singular fibers	MWL	narrow MWL	Artin invariant, $\dim \widetilde{\mathcal{X}(\mathrm{I}_l)}$
1	$\mathrm{I}_2 \times 12$	$E_8(2) \oplus \mathbb{Z}/2\mathbb{Z}$	$E_8(2)$	$9, 8$
2	$\mathrm{I}_4, \mathrm{I}_2 \times 10$	$E_7^*(2) \oplus \mathbb{Z}/2\mathbb{Z}$	$E_7(2)$	$8, 7$
3	$\mathrm{I}_6, \mathrm{I}_2 \times 9$	$E_6^*(2) \oplus \mathbb{Z}/2\mathbb{Z}$	$E_6(2)$	$7, 6$
4	$\mathrm{I}_8, \mathrm{I}_2 \times 8$	$D_5^*(2) \oplus \mathbb{Z}/2\mathbb{Z}$	$D_5(2)$	$6, 5$
5	$\mathrm{I}_{10}, \mathrm{I}_2 \times 7$	$A_4^*(2) \oplus \mathbb{Z}/2\mathbb{Z}$	$A_4(2)$	$5, 4$
6	$\mathrm{I}_{12}, \mathrm{I}_2 \times 6$	$A_2^*(2) \oplus A_1^*(2) \oplus \mathbb{Z}/2\mathbb{Z}$	$A_2(2) \oplus A_1(2)$	$4, 3$
7	$\mathrm{I}_{14}, \mathrm{I}_2 \times 5$	$\frac{2}{7}\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 8 & -2 \\ -2 & 4 \end{pmatrix}$	$3, 2$
8	$\mathrm{I}_{16}, \mathrm{I}_2 \times 4$	$\langle \frac{1}{4} \rangle \oplus \mathbb{Z}/2\mathbb{Z}$	$\langle 16 \rangle$	$2, 1$
9	$\mathrm{I}_{18}, \mathrm{I}_2 \times 3$	$\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\{0\}$	$1, 0$

E_8^4-family

Next we look into the another subfamily, $I_{0,2}^*$**-family**, which is "lower part" of non-semistable family. This gives a 4-dimensional stratification. Generically, a member of this family can be written as the following Weierstrass equation after a change of parameters.

$$y^2 + txy = x^3 + p_1^{\frac{1}{2}} tx^2 + r_2^{\frac{1}{2}} t^2 x + t^5 + r_4 t^4 + r_3 t^3$$

This generic member have a singular fiber of type $\mathrm{I}_{0,2}^*$ over $t = 0$ and four other singular fibers of type I_1.

After couple of Frobenius pullbacks they give examples for the case 1.

Remark: $\overline{\mathrm{Ig}(4)^{\mathrm{ord}}}$ and $\overline{\mathrm{Ig}(8)^{\mathrm{ord}}}$ appear in the stratification.

Frobenius tower of stratification

$$\begin{tikzcd}
\text{not } K3 \arrow[r] \arrow[d] & \text{not } K3 \arrow[r] \arrow[d] & \text{not } K3 \arrow[d] \\
\text{not } K3 \arrow[r] & \text{not } K3 \arrow[r] \arrow[d] & \text{not } K3 \arrow[d] \\
 & (\mathrm{I}^*_{0,2}, \mathrm{I}_{16}) \arrow[r] & (\mathrm{I}^*_{1,1}, \mathrm{I}_{16})
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
\text{not } K3 \arrow[r] \arrow[d] & \text{not } K3 \arrow[r] \arrow[d] & \text{not } K3 \arrow[d] \\
\text{not } K3 \arrow[r] & (\mathrm{I}^*_{0,2}, 2\mathrm{I}_8) \arrow[r] \arrow[d] & (\mathrm{I}^*_{1,1}, 2\mathrm{I}_8) \arrow[d] \\
 & (\mathrm{II}^*_2, \mathrm{I}_8) \arrow[r] & (\mathrm{III}_1, \mathrm{I}_8)
\end{tikzcd}
\Rightarrow$$

$$\begin{tikzcd}
(\mathrm{I}^*_{0,2}, 4\mathrm{I}_4) \arrow[r] \arrow[d] & (\mathrm{I}^*_{1,1}, 4\mathrm{I}_4) \arrow[r] \arrow[d] & (\mathrm{IV}^*_0, 4\mathrm{I}_4) \arrow[d] \\
(\mathrm{I}^*_{3,3}, 3\mathrm{I}_4) \arrow[r] & (\mathrm{II}^*_2, 2\mathrm{I}_4) \arrow[r] \arrow[d] & (\mathrm{III}_1, 2\mathrm{I}_4) \arrow[d] \\
 & (\mathrm{I}^*_{0,2}, \mathrm{I}_4) \arrow[r] & (\mathrm{I}^*_{1,1}, \mathrm{I}_4)
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
(\mathrm{II}^*_2, 4\mathrm{I}_2) \arrow[r] \arrow[d] & (\mathrm{III}^*_1, 4\mathrm{I}_2) \arrow[r] \arrow[d] & (\mathrm{IV}_0, 4\mathrm{I}_2) \arrow[d] \\
(\mathrm{III}^*_3, 3\mathrm{I}_2) \arrow[r] & (\mathrm{I}^*_{0,2}, 2\mathrm{I}_2) \arrow[r] \arrow[d] & (\mathrm{I}^*_{1,1}, 2\mathrm{I}_2) \arrow[d] \\
 & (\mathrm{I}^*_{2,2}, \mathrm{I}_2) \arrow[r] & (\mathrm{III}^*_1, \mathrm{I}_2)
\end{tikzcd}
\Rightarrow$$

$$\begin{tikzcd}
(\mathrm{I}^*_{0,2}, 4\mathrm{I}_1) \arrow[r] \arrow[d] & (\mathrm{I}^*_{1,1}, 4\mathrm{I}_1) \arrow[r] \arrow[d] & (\mathrm{IV}^*_0, 4\mathrm{I}_1) \arrow[d] \\
(\mathrm{I}^*_{0,3}, 3\mathrm{I}_1) \arrow[r] & (\mathrm{I}^*_{2,2}, 2\mathrm{I}_1) \arrow[r] \arrow[d] & (\mathrm{III}^*_1, 2\mathrm{I}_1) = \bar{\mathcal{E}}/\overline{\mathrm{Ig}(8)^{\mathrm{ord}}} \arrow[d] \\
 & (\mathrm{I}^*_{3,2}, \mathrm{I}_1) \arrow[r] & (\mathrm{II}^*_1, \mathrm{I}_1) = \bar{\mathcal{E}}/\overline{\mathrm{Ig}(4)^{\mathrm{ord}}}
\end{tikzcd}$$

Sections of order 8

There exists only one elliptic K3 surface $X \longrightarrow \mathbb{P}^1$ with 8-torsion section in characteristic 2 up to isomorphism. It has the following invariants:

singular fibers	σ_0	$\mathrm{MW}^\circ(X)$	$\mathrm{MW}(X)$
$\mathrm{I}^*_{1,1}, 2 \times \mathrm{I}_8$	1	$A_1(2)$	$A_1^*(2) \oplus (\mathbb{Z}/8\mathbb{Z})$

The Weierstraß equation is given by the following:

$$y^2 + t^2xy = x^3 + x + t^4.$$

In particular, it is the unique supersingular K3 surface with Artin invariant $\sigma_0 = 1$.

Sections of order 4

In characteristic 2, the classifying morphism φ for an elliptic K3 surface with 4-torsion section is finite of degree $2 \leq \deg \varphi \leq 4$. More precisely,

1. $\deg \varphi = 2$, φ is separable and $\varphi^{-1}(O)$ consists of two points, or
2. $\deg \varphi = 3$ and $\varphi^{-1}(O)$ consists of one points or two points, or
3. $\deg \varphi = 4$ and $\varphi^{-1}(O)$ consists of one point or two points with ramification index $e = 2$ (wildly ramified).

Conversely, if φ is as above then the associated elliptic fibration with 4-torsion section is a K3 surface.

Sections of order 4

Every (Shioda-)supersingular K3 surface with Artin invariant $\sigma_0 \leq 4$ in characteristic 2 possesses an elliptic fibration with 4-torsion section.
The complete list of these surfaces is given by the following table

$\deg \varphi$	singular fibers		dim	σ_0	$\mathrm{MW}^\circ(X)$	$\mathrm{MW}(X)$
3	$\mathrm{I}_{3,3}^*$	$3 \times \mathrm{I}_4$	2	3	$D_4(2)$	$D_4^*(2) \oplus \mathbb{Z}/4\mathbb{Z}$
	$\mathrm{I}_{3,3}^*$	$\mathrm{I}_8, \mathrm{I}_4$	1	2	A_3	$A_3^* \oplus \mathbb{Z}/4\mathbb{Z}$
	$\mathrm{I}_{3,3}^*$	I_{12}	0	1	A_2	$A_2^* \oplus \mathbb{Z}/4\mathbb{Z}$
4	φ separable:					
	$\mathrm{I}_{0,2}^*$	$4 \times \mathrm{I}_4$	3	4	$D_4(2)$	$D_4^*(2) \oplus \mathbb{Z}/4\mathbb{Z}$
	$\mathrm{I}_{0,2}^*$	$\mathrm{I}_8, 2 \times \mathrm{I}_4$	2	3	$A_3(2)$	$A_3^*(2) \oplus \mathbb{Z}/4\mathbb{Z}$
	$\mathrm{I}_{0,2}^*$	$\mathrm{I}_{12}, \mathrm{I}_4$	1	2	$A_2(2)$	$A_2^*(2) \oplus \mathbb{Z}/4\mathbb{Z}$
	φ inseparable but not purely inseparable:					
	$\mathrm{I}_{1,1}^*$	$2 \times \mathrm{I}_8$	0	1	$A_1(2)$	$A_1^*(2) \oplus \mathbb{Z}/8\mathbb{Z}$
	φ purely inseparable:					
	$\mathrm{I}_{1,1}^*$	I_{16}	0	1	$\{0\}$	$\mathbb{Z}/4\mathbb{Z}$

Elliptic $K3$ surfaces with p^n-torsion sections

p^n	$[\mathrm{Ig}(p^n)^{\mathrm{ord}}]$	$\deg \varphi$	description of the family
$8 = 2^3$	fine	1	unique supersingular $K3$ ($\sigma_0 = 1$)
7	fine	1	unique supersingular $K3$ ($\sigma_0 = 1$)
5	fine	2	2-dim ordinary $K3$'s $\supset$ 1-dim s.s. $K3$'s ($\sigma_0 \leq 2$)
4	fine	$2, 3, 4$	4-dim ordinary $K3$'s $\supset$ 3-dim s.s. $K3$'s ($\sigma_0 \leq 4$)
3	fine	$2, \cdots, 6$	6-dim ordinary $K3$'s $\supset$ 5-dim s.s. $K3$'s ($\sigma_0 \leq 6$)
2	not fine		isotrivial case: Kummer surfaces; non-isotrivial case: Many elliptic surfaces have 2-torsions

THE END