DEHN TWIST PRESENTATIONS OF FINITE GROUP ACTIONS ON THE ORIENTED SURFACES OF GENUS 4

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1. Introduction

Let Σ_g be a closed oriented surface of genus g, and $\mathrm{Homeo}_+(\Sigma_g)$ a group of orientation preserving homeomorphisms over Σ_g. In this note, for two elements f_1, f_2 $\in \mathrm{Homeo}_+(\Sigma_g)$, $f_1 f_2$ means applying f_1 and then f_2. The group $\mathcal{M}(\Sigma_g)$ of isotopy classes of $\mathrm{Homeo}_+(\Sigma_g)$ is called the *mapping class group* of Σ_g. Any finite subgroup of $\mathcal{M}(\Sigma_g)$ can be considered as the automorphism group of some algebraic curve, and hence finite subgroups of $\mathcal{M}(\Sigma_g)$ are investigated in various contexts. On the other

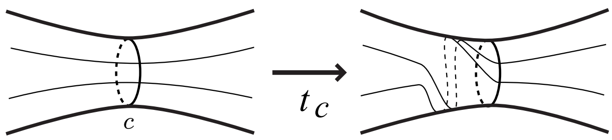


Figure 1

hand, Dehn [3] and Lickorish [10] proved that $\mathcal{M}(\Sigma_g)$ is generated by *Dehn twists*. For a simple closed curve c on Σ_g, the homeomorphism t_c on Σ_g indicated in Figure 1 is called the Dehn twist about c. Since actions on Dehn twists on geometric objects on Σ_g, for example homology groups of Σ_g, are easy to understand, presentations of elements in $\mathcal{M}(\Sigma_g)$ by Dehn twists are useful for the investigation on $\mathcal{M}(\Sigma_g)$. For periodic elements in $\mathcal{M}(\Sigma_g)$, Ishizaka [7] completely obtained Dehn twist presentations in hyperelliptic case, and the author [5] obtained Dehn twist presentations when g is at most 4. Nakamura-Nakanishi [11] (resp. the author [6]) obtained Dehn twist presentations of finite groups actions on Σ_2 (resp. on Σ_3). In this note, we investigate on Dehn twist presentations of finite group actions on Σ_4.

2. Maximal finite group actions over Σ_4

An injection ϵ from a finite group $\mathcal{G}$ to $\mathrm{Homeo}_+(\Sigma_g)$ is called the *action of* $\mathcal{G}$ *over* Σ_g. In this paper, the group $\mathcal{G}$ acts on Σ_3 from the right; the action of $g \in \mathcal{G}$ on $x \in \Sigma_3$ is written as $x\epsilon(g)$ or xg. For a system of generators $\{g_1, \ldots, g_k\}$ of $\mathcal{G}$, we call a

This research is supported by Grant-in-Aid for Scientific Research (C) (No. 16K05156), Japan Society for the Promotion of Science.

system of Dehn twist presentations of the isotopy classes $\epsilon(g_1), \ldots, \epsilon(g_k)$ a *Dehn twist presentation for the finite group action* $\epsilon : \mathcal{G} \to \mathrm{Homeo}_+(\Sigma_g)$. Two finite group actions $\epsilon_1, \epsilon_2 : \mathcal{G} \to \mathrm{Homeo}_+(\Sigma_g)$ are *equivalent* if there is an automorphism ω of $\mathcal{G}$ and an orientation preserving homeomorphism h over Σ_g which satisfy $\epsilon_2(g) = h^{-1}\epsilon_1(\omega(g))h$ for any $g \in \mathcal{G}$.

For an action of a finite group $\mathcal{G}$ over Σ_g, $\epsilon : \mathcal{G} \to \mathrm{Homeo}_+(\Sigma_g)$ and a subgroup $\mathcal{H}$ of $\mathcal{G}$, we can define an action of $\mathcal{H}$ over Σ_g by the restriction $\epsilon|_{\mathcal{H}} : \mathcal{H} \to \mathrm{Homeo}_+(\Sigma_g)$ and we call this action a *subgroup action* of ϵ. If we obtain a Dehn twist presentation of a action ϵ of $\mathcal{G}$ over Σ_g, then we obtain a Dehn twist presentation of a subgroup action $\epsilon|_{\mathcal{H}}$ automatically. Therefore, we will obtain Dehn twist presentations of maximal finite group actions over Σ_3.

We remark here that we checked Dehn twist presentations by using T4M7 * implemented by K. Ahara, T. Sakasai, M. Suzuki.

Bogopol'skiĭ [2] obtained a complete list of finite group actions on Σ_4 and showd:

Theorem 1. [2] *Any finite group action on Σ_4 is a subgroup action of the actions of following groups:*

$$\mathbb{Z}_{15}, \mathbb{Z}_{18}, SL_2(3), S_3 \times \mathbb{Z}_6, S_4 \times \mathbb{Z}_3, S_5, D_{2,16,7}, \mathbb{Z}_5 \rtimes D_4, (Z_3 \times Z_3) \rtimes D_4.$$

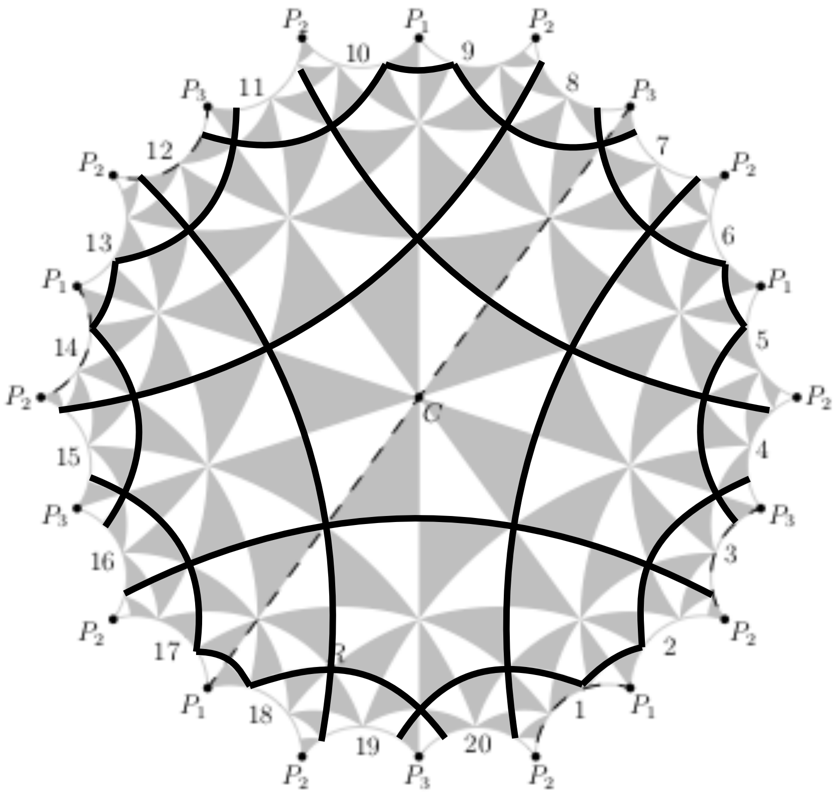
Remark 2. 1. In general, for a finite group G, its action on Σ_g is not unique. Nevertheless, for each of the above nine groups, its action on Σ_3 is unique up to equivalence.
2. The actions of $\mathbb{Z}_{15}$, $\mathbb{Z}_{18}$ are generated by $f_{4,5}$ and $f_{4,1}$ in [5, Theorem 3.1] and Dehn twist presentations of them are obtained in [5, Theorem 3.2].
3. The actions of $SL_2(3)$, $D_{2,16,7}$, $\mathbb{Z}_5 \ltimes D_4$ commute with the hyperelliptic involution, and the Dehn twist presentations for these actions are investigated by Hasegawa [4].
4. For the action of S_5 on Σ_4, we already obtained a Dehn twist presentation (§2.1). For the actions of $S_4 \times \mathbb{Z}_3$, $\mathbb{Z}_5 \rtimes D_4$, $(Z_3 \times Z_3) \rtimes D_4$, the author is trying to find Dehn twist presentations. In §2.2, 2.3, 2.4, we show the pictures of these actions which would be useful to find Dehn twist presentations.

2.1. **The action of S_5 on Σ_4 and its Dehn twist presentation.** Bring's curve is an complex curve of genus 4 which is a complete intersection of three hypersurfaces in $\mathbb{C}P^4$ defined by:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 0, \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0, \\ x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0. \end{cases}$$

On this curve, S_5 acts as the permutation of coordinates $\{x_1, x_2, x_3, x_4, x_5\}$. Riera-Rodríguez [14] obtained a hyperbolic model of Bring's curve as shown in Figure 2.

* Downloadable from http://www.ms.u-tokyo.ac.jp/˜sakasai/MCG/MCG.html



Edge identifications

$1 \leftrightarrow 14$	$5 \leftrightarrow 18$	$9 \leftrightarrow 2$	$13 \leftrightarrow 6$	$17 \leftrightarrow 10$
$3 \leftrightarrow 12$	$7 \leftrightarrow 16$	$11 \leftrightarrow 20$	$15 \leftrightarrow 4$	$19 \leftrightarrow 8$

FIGURE 2

If we identify the pairs the edges on the boundary of 20-gon as indicated in this figure, we have a picture of Σ_4 as shown in Figure 3. In these figures, the pentagons with thick lines on Figure 2 correspond to the pentagons on Figure 3. There are 24 pentagons in each figure, and these in Figure 3 have vertices indexed by $\{1, 2, 3, 4, 5\}$ and each pentagon corresponds to a circular arrangement of $\{1, 2, 3, 4, 5\}$. The group S_5 acts on the set of these arrangements and this defines the action of S_5 on Σ_4, for example, the transposition $(1, 2)$ sends the pentagon $(1, 2, 3, 4, 5)$ to the pentagon $(2, 1, 3, 4, 5) = (1, 3, 4, 5, 2)$ and so on. Cyclic permutations $a = (5, 4, 3, 2, 1)$ and $d = (2, 3, 4, 5)$ generates S_5. The action of d on Σ_4 is the clockwise $\pi/4$ rotation of Figure 3. This periodic map is called *propella* by Okuda-Takamura [13] and the Dehn twist presentation of this map was obtained by them. The element a of S_5 acts on Σ_4 as a periodic map of order 5 and, on the right bottom of Figure 4, sends the arcs $0 \to 1 \to 2 \to 3 \to 4 \to 0$ and fixes the thick circle setwisely. This periodic map is $f_{4,12}$ in [5, Proposition 3.1]. A Dehn twist presentation for $f_{4,12}$ was obtained in [5,

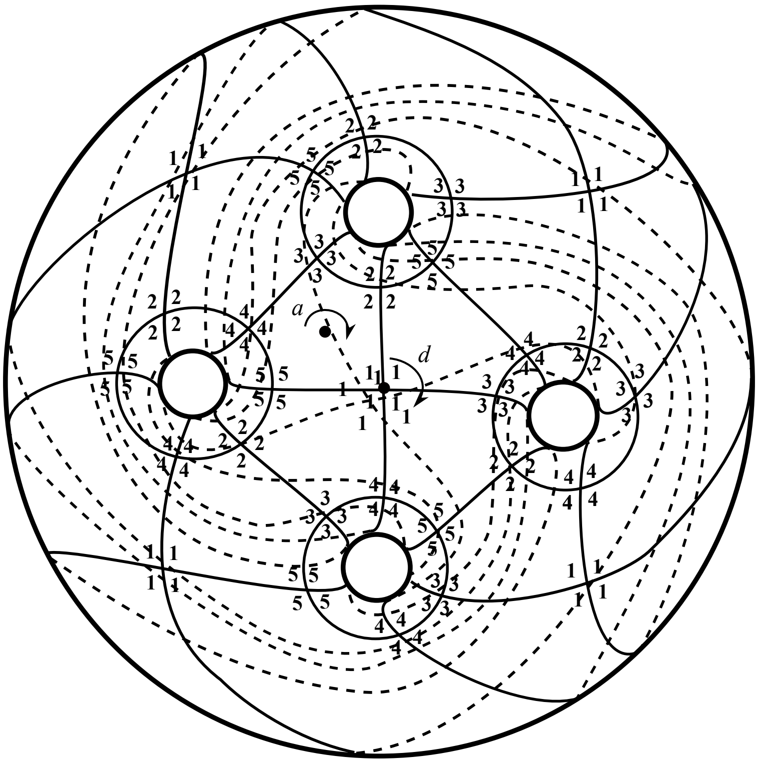


FIGURE 3

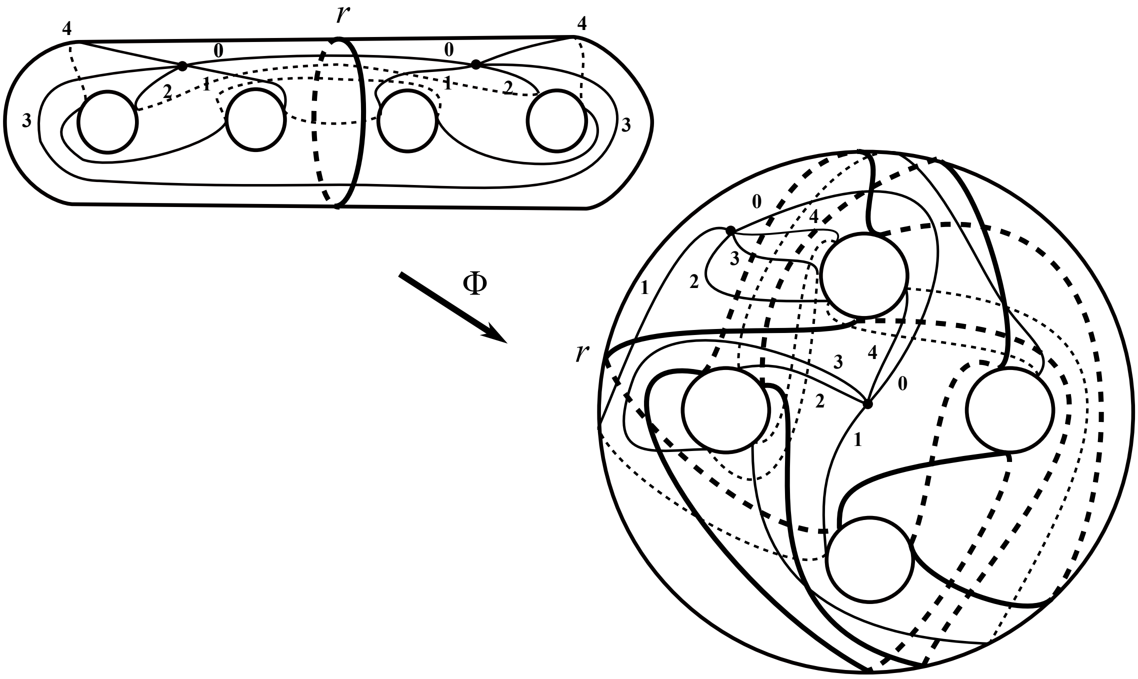


FIGURE 4

Proposition 3.1]. For the circles c_i in Figure 5,

$$t_{c_2}t_{c_3}t_{c_4}t_{c_{12}}t_{c_3}t_{c_3}t_{c_{10}}t_{c_3}t_{c_8}^{-1}t_{c_7}^{-1}t_{c_6}^{-1}t_{c_{13}}^{-1}t_{c_7}^{-1}t_{c_6}^{-1}t_{c_{11}}^{-1}t_{c_7}^{-1}$$

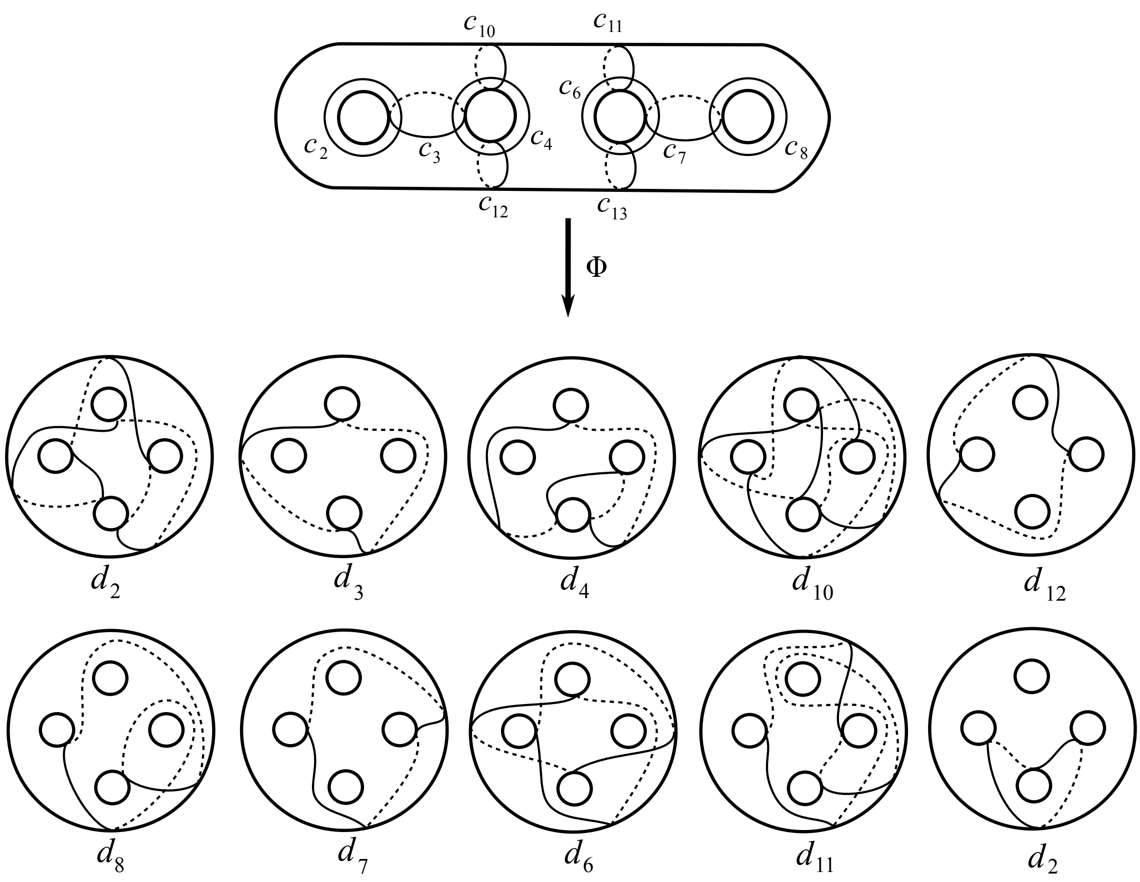


FIGURE 5

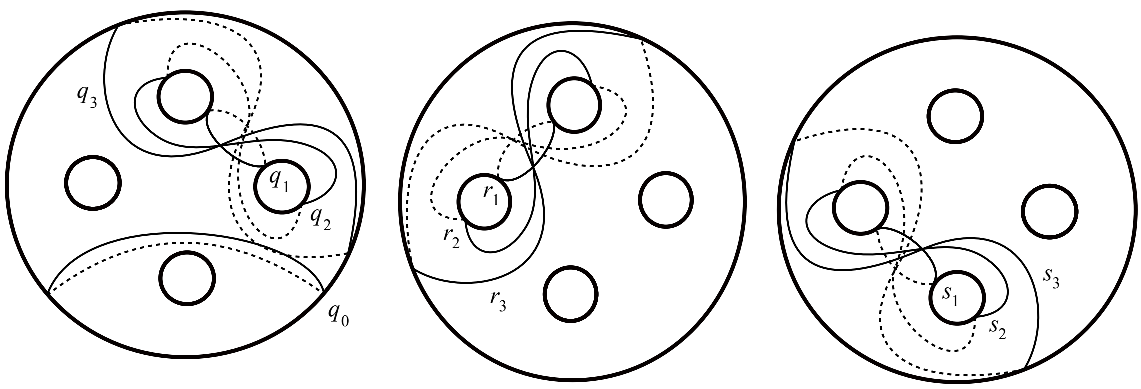


FIGURE 6

is $f_{4,12}$ which sends the arcs $0 \to 1 \to 2 \to 3 \to 4 \to 0$ and fixes the thick circle setwisely on the upper left of Figure 4. Let Φ be a homomorphism from the upper left to the lower right of Figure 4 which sends arcs with same numbers and the thick circles and $d_i = \Phi(c_i)$ in Figure 5, then

$$a = t_{d_2}t_{d_3}t_{d_4}t_{d_{12}}t_{d_3}t_{d_3}t_{d_{10}}t_{d_3}t_{d_8}^{-1}t_{d_7}^{-1}t_{d_6}^{-1}t_{d_{13}}^{-1}t_{d_7}^{-1}t_{d_6}^{-1}t_{d_{11}}^{-1}t_{d_7}^{-1}$$

In summary, we have:

Proposition 3. *The action of* $S_5 = \langle a, d \,|\, a^5 = d^4 = 1, (ad)^2 = 1 \rangle$ *on* Σ_4 *is generated by* $a = t_{d_2}t_{d_3}t_{d_4}t_{d_{12}}t_{d_3}t_{d_3}t_{d_{10}}t_{d_3}t_{d_8}^{-1}t_{d_7}^{-1}t_{d_6}^{-1}t_{d_{13}}^{-1}t_{d_7}^{-1}t_{d_6}^{-1}t_{d_{11}}^{-1}t_{d_7}^{-1}$, $d = t_{q_1}t_{q_2}t_{q_3}t_{r_1}t_{r_2}t_{r_3}t_{s_1}t_{s_2}t_{s_3}t_{q_0}$. *In the above presentation,* d_i, q_i, r_i, s_i *are simple closed curves shown in Figure 5 and Figure 6.*

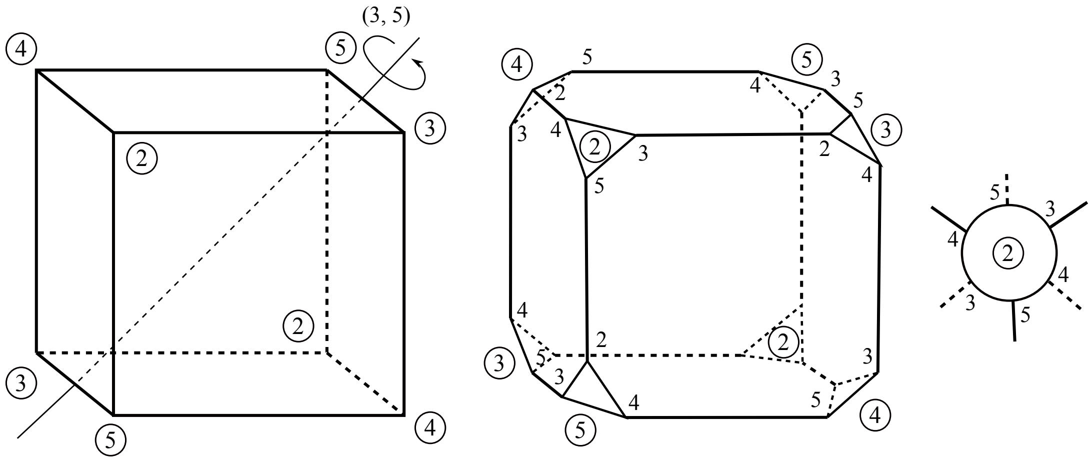


FIGURE 7

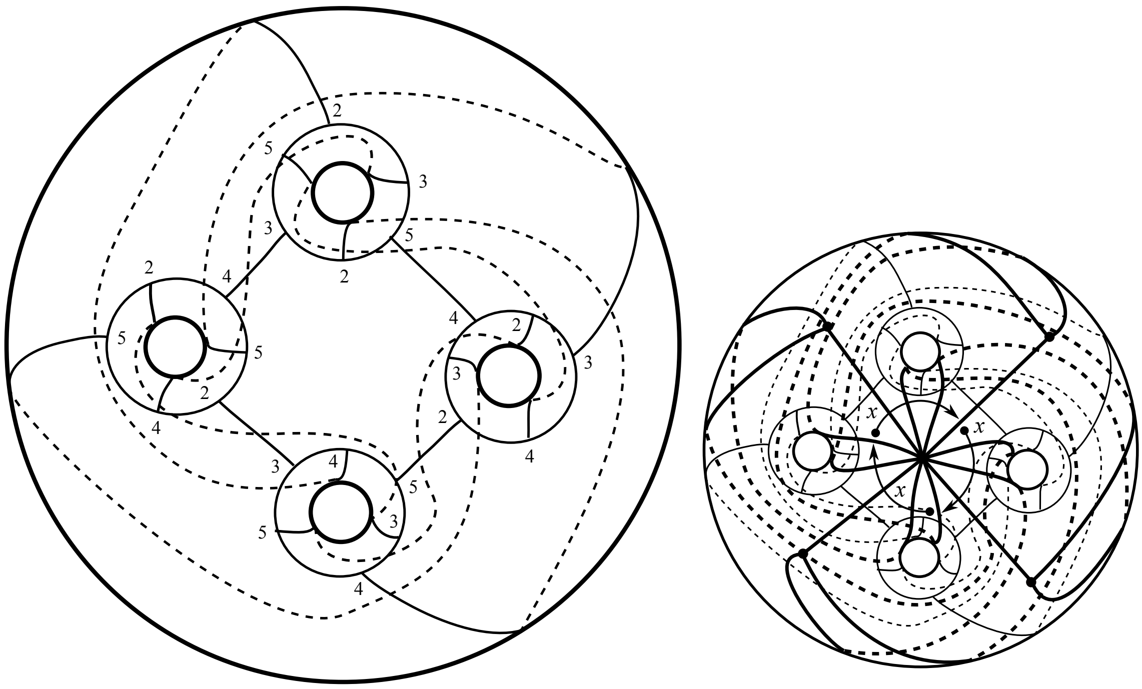


FIGURE 8

2.2. **The action of** $S_4 \times \mathbb{Z}_3$ **on** Σ_4**.** In this subsection, we regard S_4 to be the subgroup of S_5 each element of which fixes 1. This group acts naturally on the cube. See Figure

7. We put the numbers (2), ..., (5) on vertices so that the antipodal vertices have the same number. The π-rotation about the axis through the middle points of edges (i) (j) is the action of the transposition $(i, j) \in S_4$. We truncate this cube at vertices then we have 8 cycles correspond to the vertices. Further we put numbers to new vertices by: for an edge begins from the cycle corresponding to (i) to the cycle corresponding to (j), we put j to the start point and i to the terminal point. See the middle of Figure 7. We glue the antipodal cycles obtained as a truncation of the vertices with the same number as shown in the right of Figure 7, where we show the case where the numbering is (2). Then we have the left of Figure 8. We remark that if we divide each face into 4 pentagons then we have Figure 3. The right of Figure 8 indicates the action of x which is a periodic map of period 3 and commutes with the action S_4 on Σ_4.

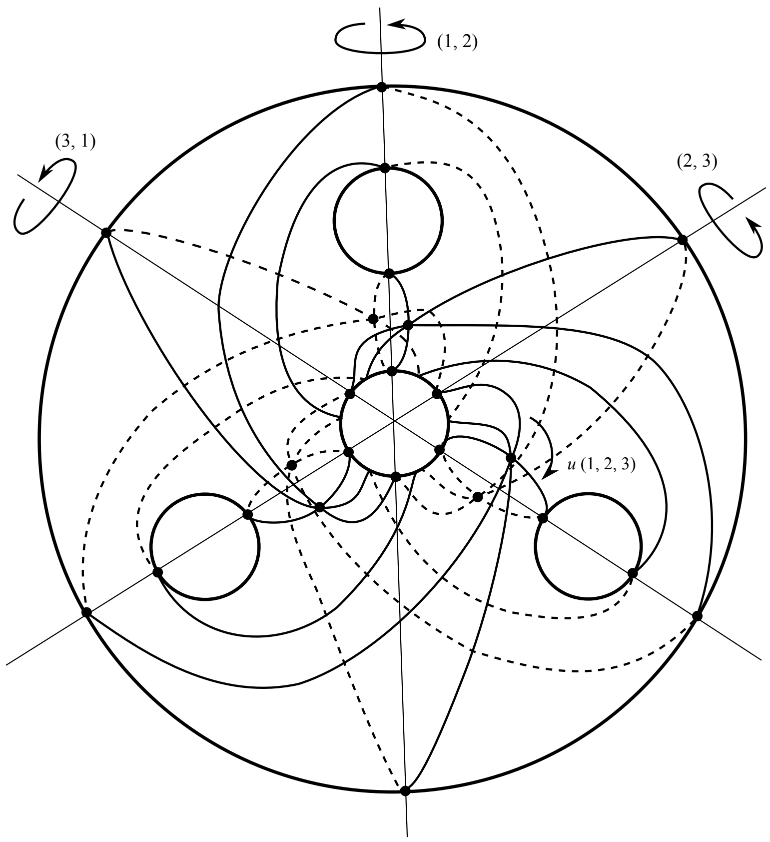


FIGURE 9

2.3. **The action of $S_3 \times \mathbb{Z}_6$ on Σ_4.** Let $S_3 \times \mathbb{Z}_6 = S_3 \times \langle u \mid u^6 = 1 \rangle$, $q : \Sigma_4 \to \Sigma_4 / S_3 \times \mathbb{Z}_6$ the quotient map, p_1 a point in Σ_4 whose isotropy group is $\mathbb{Z}_2$ generated by $(1, 2)$ and p_2 a point in Σ_4 whose isotropy group is $\mathbb{Z}_6$ generated by $u(1, 2, 3)$. The graph in Figure 9 is the inverse image by q of the arc in $\Sigma_4 / S_3 \times \mathbb{Z}_6$ connecting $q(p_1)$ and $q(p_2)$.

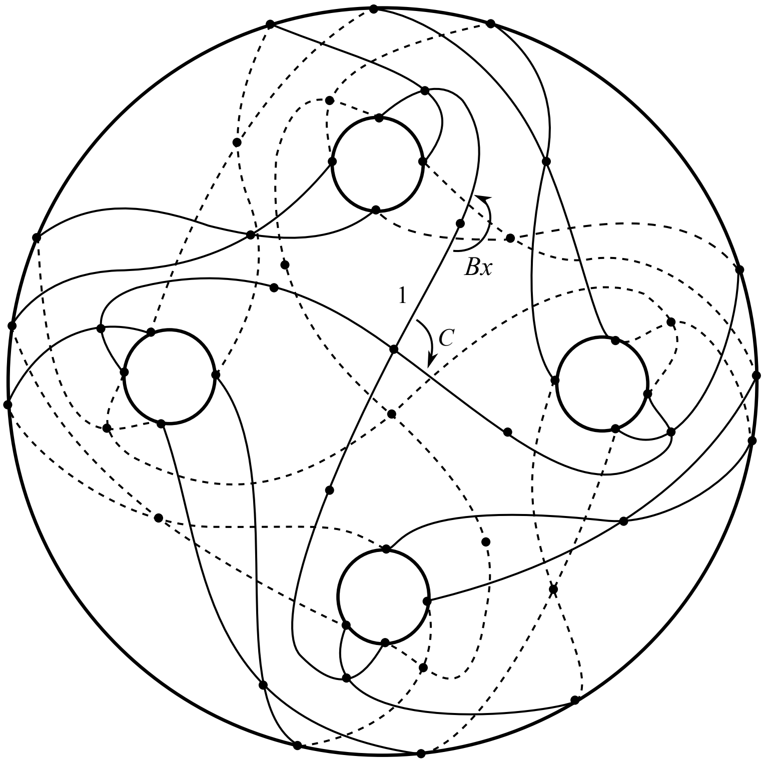


FIGURE 10

2.4. **The action of** $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes D_4$ **on** Σ_4**.** Let

$$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes D_4 = \left\langle x, y, B, C \;\middle|\; \begin{array}{c} x^3 = y^3 = 1,\, xy = yx,\, B^2 = C^4 = (BC)^2 = 1 \\ Cx = yC,\, Cy = x^{-1}C,\, Bx = x^{-1}B,\, By = yB \end{array} \right\rangle,$$

$q' : \Sigma_4 \to \Sigma_4/(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes D_4$ the quotient map, p'_1 a point in Σ_4 whose isotropy group is $\mathbb{Z}_4$ generated by C and p'_2 a point in Σ_4 whose isotropy group is $\mathbb{Z}_2$ generated by xB. The graph in Figure 10 is the inverse image by q' of the arc in $\Sigma_4/(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes D_4$ connecting $q'(p'_1)$ and $q'(p'_2)$.

Acknowledgement The author thanks Professors Akira Ohbuchi and Jiryo Komeda, the organizers of the 16-th Symposium on Algebraic Curves.

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