DEHN TWIST PRESENTATIONS OF FINITE GROUP ACTIONS ON THE ORIENTED SURFACES OF GENUS 3

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Abstract. In this note we give presentations of all finite subgroups of the mapping class group of a closed surface of genus 3 by Dehn twists up to conjugacy.

1. Introduction

Let Σ_g be a closed oriented surface of genus g, and $\mathrm{Homeo}_+(\Sigma_g)$ a group of orientation preserving homeomorphisms over Σ_g. In this paper, for two elements f_1, f_2 $\in \mathrm{Homeo}_+(\Sigma_g)$, $f_1 f_2$ means appling f_1 and then f_2. The group $\mathcal{M}(\Sigma_g)$ of isotopy classes of $\mathrm{Homeo}_+(\Sigma_g)$ is called the *mapping class group* of Σ_g. Any finite subgroup of $\mathcal{M}(\Sigma_g)$ can be considered as the automorphism group of some algebraic curve, and hence finite subgroups of $\mathcal{M}(\Sigma_g)$ are investigated in various contexts. On the other

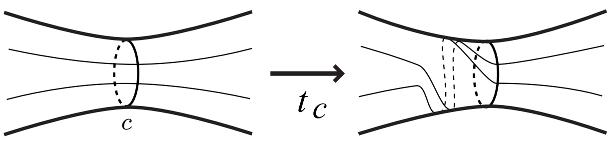

Figure 1

hand, Dehn [3] and Lickorish [11] proved that $\mathcal{M}(\Sigma_g)$ is generated by *Dehn twists*. For a simple closed curve c on Σ_g, the homeomorphism t_c on Σ_g indicated in Figure 1 is called the Dehn twist about c. Since actions on Dehn twists on geometric objects on Σ_g, for example homology groups of Σ_g, are easy to understand, presentations of elements in $\mathcal{M}(\Sigma_g)$ by Dehn twists are useful for the investigation on $\mathcal{M}(\Sigma_g)$. For periodic elements in $\mathcal{M}(\Sigma_g)$, Ishizaka [6] completely obtained Dehn twist presentations in hyperelliptic case, and the author [5] obtained Dehn twist presentations when g is at most 4. Nakamura-Nakanishi [12] obtained Dehn twist presentations of finite groups actions on Σ_2. In this note, we investigate on Dehn twist presentations of finite group actions on Σ_3.

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2. Dehn twist presentations for the periodic maps over Σ_3

At first, we explain the classification of periodic map over Σ_g by Nielsen [13]. A homeomorphism f over Σ_g is called *periodic* if there is an integer $n \geq 1$ which satisfies $f^n = id_{\Sigma_g}$. The least integer n which satisfies the above condition is called the *period* of f. Let n be the period of periodic map f over Σ_g. A point p on Σ_g is a *multiple point* if there is an integer k such that $0 < k < n$ and $f^k(p) = p$, and $M_f \subset \Sigma_g$ denotes the set of multiple points of f. Let Σ_g/f be the *orbit space* of f, and $\pi_f : \Sigma_g \to \Sigma_g/f$ the quotient map, i.e. a map defined by $\pi_f(x) = [x]$, where $[x]$ is the point on Σ_g/f represented by the point $x \in \Sigma_g$. Then π_f is an n-fold branched covering ramified at $\pi_f(M_f) \subset \Sigma_g/f$. Hence, we call $B_f = \pi_f(M_f)$ a set of *branch point*. In the above situation, $\pi_f|_{\Sigma_g \setminus M_f} : \Sigma_g \setminus M_f \to (\Sigma_g/f) \setminus B_f$ is an n-fold covering in an ordinary meaning.

We define a homomorphism $\Omega_f : \pi_1((\Sigma_g/f) \setminus B_f, x) \to \mathbb{Z}_n$ describing this n-fold covering $\pi_f|_{\Sigma_g \setminus M_f}$ as follows. We choose a point $\tilde{x}$ in Σ_g such that $\pi_f(\tilde{x}) = x$. Let $l : [0,1] \to (\Sigma_g/f) \setminus B_f$ be a loop satisfying $l(0) = l(1) = x$. We make a lift $\tilde{l} : [0,1] \to \Sigma_g$ of l which begins from $\tilde{l}(0) = \tilde{x}$, then we see $\pi_f(\tilde{l}(1)) = l(1) = x$, i.e., $\tilde{l}(1)$ is in the orbit of $\tilde{l}(0) = \tilde{x}$ by the periodic map f, hence there is an integer k such that $f^k(\tilde{x}) = \tilde{l}(1)$. We set $\Omega_f([l]) = k \in \mathbb{Z}_n$. Since $\mathbb{Z}_n$ is an Abelian group, this homomorphism Ω_f induces a homomorphism $\omega_f : H_1((\Sigma_g/f) \setminus B_f) \to \mathbb{Z}_n$.

Two periodic maps f, f' over Σ_g are *conjugate* if there is a homomorphism g from Σ_g to itself such that $f' = g \circ f \circ g^{-1}$.

For each branch point $Q_i \in B_f$ of f, we choose mutually disjoint small disk D_{Q_i} and orient the boundary S_{Q_i} of D_{Q_i} clockwisely.

Theorem 1. [13] *Two periodic maps f, f' over Σ_g are conjugate if and only if the following three conditions are satisfied,*
(1) the period of f = the period of f',
(2) the number of elements of B_f = the number of elements of $B_{f'}$,
(3) if we change the numbering of $B_{f'} = \{Q'_1, Q'_2, \cdots, Q'_b\}$ properly, we have $\omega_f(S_{Q_i}) = \omega_{f'}(S_{Q'_i})$ for each i. □

By the above theorem, the notation $\left(n, \frac{\theta_1}{n} + \cdots + \frac{\theta_b}{n}\right)$, where n = the period of f, $\theta_i = \omega_f(S_{Q_i})$, completely determines the conjugacy class of f. This notation is introduced by Ashikaga and Ishizaka [1] and called *total valency*.

Dehn twist presentations of periodic maps over Σ_3 are obtained as follows.

Theorem 2. [5] *Any periodic map on Σ_3 is a power of some of the following periodic maps,*

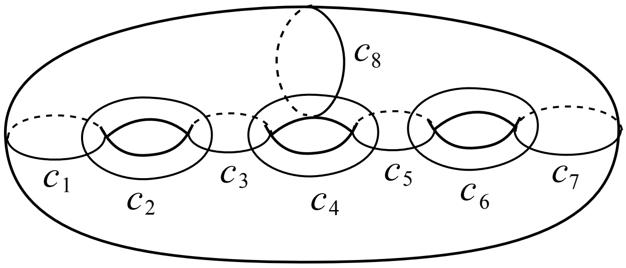

FIGURE 2

$$f_{3,1} = \left(14, \frac{1}{14} + \frac{3}{7} + \frac{1}{2}\right),\ f_{3,2} = \left(12, \frac{1}{12} + \frac{5}{12} + \frac{1}{2}\right),\ f_{3,3} = \left(8, \frac{1}{8} + \frac{1}{8} + \frac{3}{4}\right),$$
$$f_{3,4} = \left(4, \frac{1}{2} + \frac{1}{2}\right),\ f_{3,5} = \left(2, \right),\ f_{3,6} = \left(12, \frac{1}{12} + \frac{1}{4} + \frac{2}{3}\right),\ f_{3,7} = \left(8, \frac{1}{8} + \frac{1}{4} + \frac{5}{8}\right),$$
$$f_{3,8} = \left(9, \frac{1}{9} + \frac{1}{3} + \frac{5}{9}\right),\ f_{3,9} = \left(7, \frac{1}{7} + \frac{2}{7} + \frac{4}{7}\right).$$

Up to conjugacy, these periodic maps are presented as products of Dehn twists as follows. In the following presentatins k means the Dehn twist about the simple closed curve c_k in Figure 2.

$$f_{3,1} = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1, \quad f_{3,2} = 6 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1, \quad f_{3,3} = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1,$$
$$f_{3,4} = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot (7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)^3,$$
$$f_{3,5} = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot (7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)^5,$$
$$f_{3,6} = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 8, \quad f_{3,7} = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 5 \cdot 4 \cdot 3 \cdot 8, \quad f_{3,8} = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 8,$$
$$f_{3,9} = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 5 \cdot 4 \cdot 8.$$

3. MAXIMAL FINITE GROUP ACTIONS OVER Σ_3 AND THEIR DEHN TWIST PRESENTATIONS

An injection ϵ from a finite group $\mathcal{G}$ to $\mathrm{Homeo}_+(\Sigma_g)$ is called the *action of* $\mathcal{G}$ *over* Σ_g. In this paper, the group $\mathcal{G}$ acts on Σ_3 from the right; the action of $g \in \mathcal{G}$ on $x \in \Sigma_3$ is written as $x\epsilon(g)$ or xg. For a system of generators $\{g_1, \ldots, g_k\}$ of $\mathcal{G}$, we call a system of Dehn twist presentations of the isotopy classes of $\{\epsilon(g_1), \ldots, \epsilon(g_k)\}$ a *Dehn twist presentation for the finite group action* $\epsilon : \mathcal{G} \to \mathrm{Homeo}_+(\Sigma_g)$. Two finite group actions $\epsilon_1, \epsilon_2 : \mathcal{G} \to \mathrm{Homeo}_+(\Sigma_g)$ are *equivalent* if there is an automorphism ω of $\mathcal{G}$ and an orientation preserving homeomorphism h over Σ_g which satisfy $\epsilon_2(g) = h^{-1}\epsilon_1(\omega(g))h$ for any $g \in \mathcal{G}$.

For an action of a finite group $\mathcal{G}$ over Σ_g, $\epsilon : \mathcal{G} \to \mathrm{Homeo}_+(\Sigma_g)$ and a subgroup $\mathcal{H}$ of $\mathcal{G}$, we can define an action of $\mathcal{H}$ over Σ_g by the restriction $\epsilon|_{\mathcal{H}} : \mathcal{H} \to \mathrm{Homeo}_+(\Sigma_g)$ and we call this action a *subgroup action* of ϵ. If we obtain a Dehn twist presentation of a action ϵ of $\mathcal{G}$ over Σ_g, then we obtain a Dehn twist presentation of a subgroup action $\epsilon|_{\mathcal{H}}$ automatically. Therefore, we will obtain Dehn twist presentations of maximal finite group actions over Σ_3.

We remark here that we checked Dehn twist presentations by using T4M7 * implemented by K. Ahara, T. Sakasai, M. Suzuki.

Based on the classification of finite group actions on Σ_3 by Broughton [2], we see:

Proposition 3. *Any finite group action on Σ_3 is a subgroup action of the actions of following groups:*

$$\mathbb{Z}_9, \mathbb{Z}_{14}, D_{2,12,5}, \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_8), \mathbb{Z}_2 \times S_4, \mathbb{Z}_2 \ltimes SL_2(3), S_3 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4), PSL_2(7).$$

Remark 4. 1. In general, for a finite group G, its action on Σ_g is not unique. Nevertheless, for each of the above eight groups, its action on Σ_3 is unique up to equivalence.
2. In §3, we make a list of other finite group actions as subgroup actions of the above eight finite group actions.
3. On Broughton's list, there is no action of $\mathbb{Z}_2 \ltimes SL_2(3)$.
4. The actions of $\mathbb{Z}_9$, $\mathbb{Z}_{14}$ are generated by $f_{3,8}$ and $f_{3,1}$ in Theorem 2 respectively and Dehn twist presentations of them are obtained in this theorem.

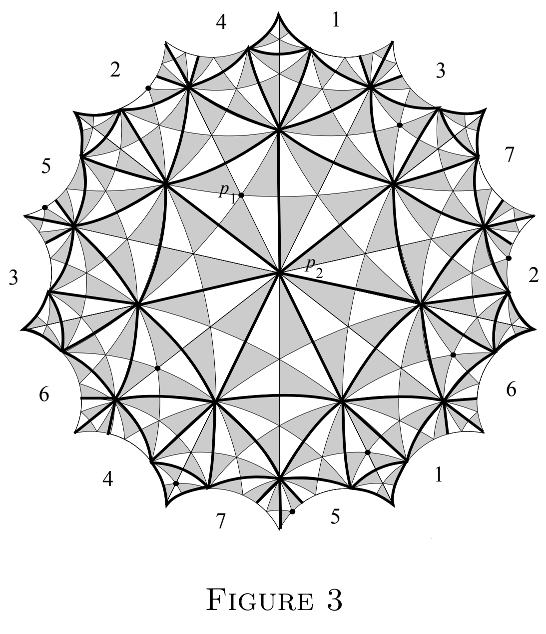

FIGURE 3

3.1. **A Dehn twist presentation of the action of $PSL_2(7)$.** By Hurwitz, it was shown that orders of finite subgroups of $\mathcal{M}(\Sigma_g)$ are at most $84(g-1)$. When $g = 3$ there is a subgroup of $\mathcal{M}(\Sigma_3)$ whose order is $84(3-1) = 168$. This subgroup is the

* Downloadable from http://www.ms.u-tokyo.ac.jp/˜sakasai/MCG/MCG.html

automorphism group of the Klein quartic $\{(x : y : z) \in \mathbb{C}P^2 \,|\, x^3y + y^3z + z^3y = 0\}$, and is isomorphic to

$$PSL_2(7) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{Z}_7, ad - bc = 1 \right\} / \{\pm E_2\}.$$

Figure 3 explains the action of $PSL_2(7)$ on Σ_3. Let G be the clockwise $2\pi/3$ rotation whose center is p_1 and F the clockwise $2\pi/7$ rotation whose center is p_2. Then the action of $PSL_2(7)$ is generated by F and G, and the relations are $F^7 = G^3 = (GF)^2 = (GFG^{-1}F^{-1})^4 = 1$. Figure 4 is obtained from Figure 3 by identifying each pair of edges

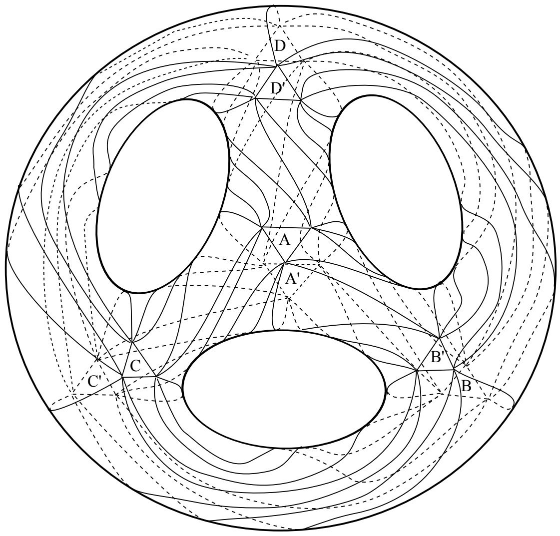

FIGURE 4

with the same number. The triangles in Figure 4 correspond to the thick triangles in Figure 3; for example, the triangle with a symbol "A" is a triangle obtained from six triangles encircling the point p_1. In order to explain the way how to obtain Figure 4 in detail, we explain the action of G. In Figure 5, G fixes A and A', and sends other points as $B \to C' \to D$, $B' \to C \to D'$. Let L_A be the union of two thick triangles which are the union of triangles surrounding A and A' and the three arcs connecting these thick triangles. We obtain T_B, T_C and T_D in the same way as above. If we remove $T_A, \ldots, T_D$ from Σ_3, we have six annuli $I, \cdots, IV$ remained. Each of these annuli are divided into eight thick triangles. In the left of Figure 6, we put $T_A, \ldots, T_D$ on Σ_3 such that G acts as a $2\pi/3$ rotation. The annuls I is as in the middle of Figure 6. We put this triangulated annulus with taking care of the edge correspondence, then we have the right of Figure 6. After we put the other annuli $II, \cdots, VI$ in the same way, we have Figure 4.

In Figure 4 , G is a $2\pi/3$ rotation about the center of A, and is a periodic map with order 3 whose Dehn twist presentation was obtained by Okuda-Takamura [14]. On

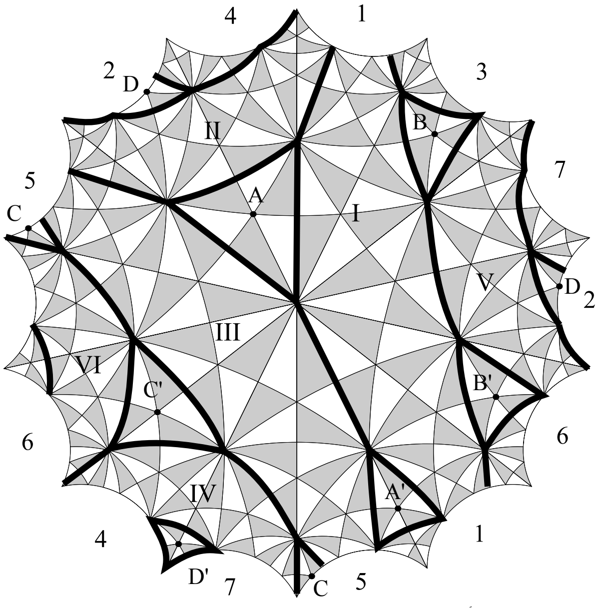

FIGURE 5

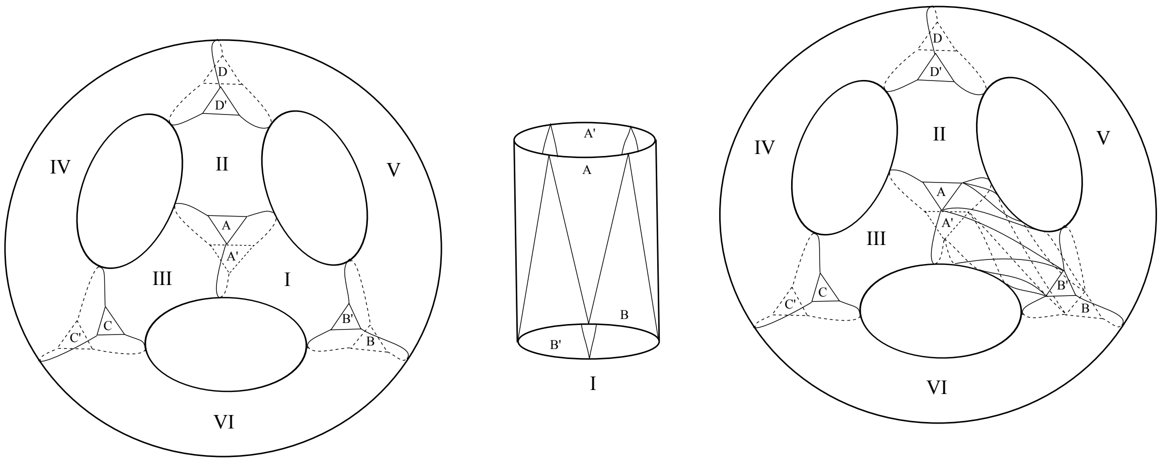

FIGURE 6

the left of Figure 7, F exchanges edges $1, 2, \cdots, 7$ clockwisely. If we draw these edges $1, 2, \cdots, 7$ on Figure 4, we obtain the right of Figure 7. By this figure, we can observe that the total valency of F is $\left(7, \frac{1}{7} + \frac{2}{7} + \frac{4}{7}\right)$, and F corresponds to $f_{3,9}$ in Theorem 2. The product of Dehn twists $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 5 \cdot 4 \cdot 8$ representing $f_{3,9}$ brings the edges on a figure of Σ_3 on the upper left of Figure 8 according to their numbering. The upper left of Figure 8 is excerpted from Figure 21 of [5]. Let Φ be an orientation preserving homeomorphism from the upper left Σ_3 to the lower left Σ_3 in Figure 8 which brings the arcs with the same number. Since the complement of these arcs is an open disk in

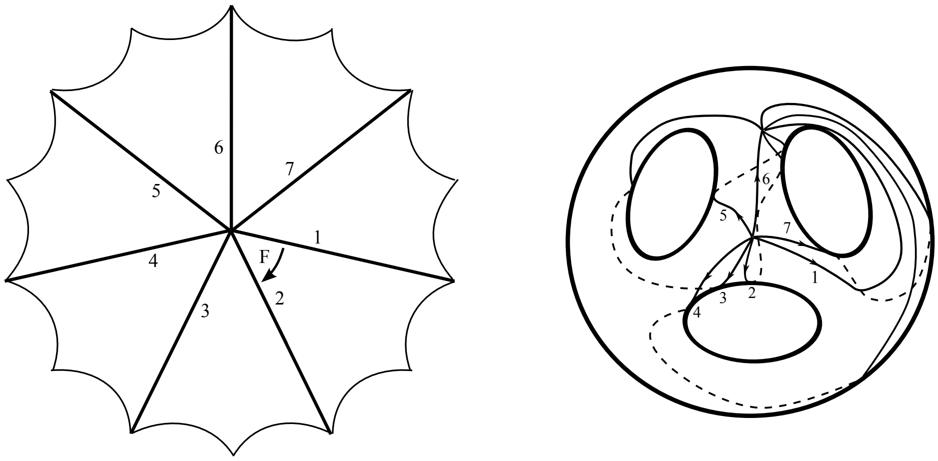

FIGURE 7

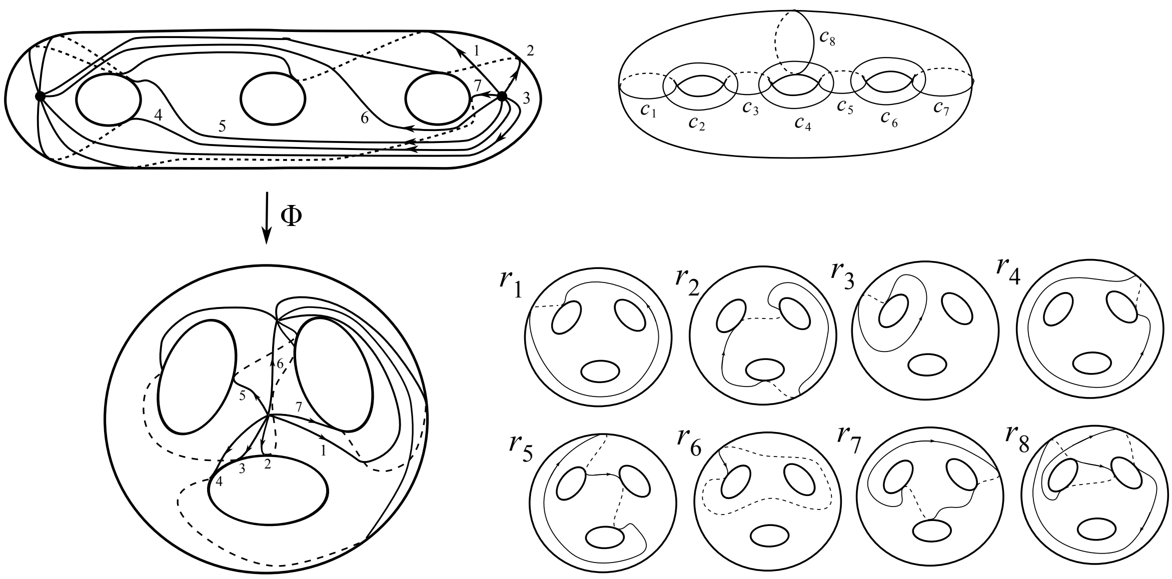

FIGURE 8

each Σ_3, Φ is uniquely determined up to isotopy. The circles $r_i = \Phi(c_i)$ are as in the lower right of Figure 8, and we see $F = t_{r_6}t_{r_5}t_{r_4}t_{r_3}t_{r_2}t_{r_1}t_{r_5}t_{r_4}t_{r_8}$.

Proposition 5. *The automorphism group* $PSL_2(7)$ *of Klein quartic* $\{(x : y : z) \in \mathbb{C}P^2 \,|\, x^3y + y^3z + z^3y = 0\}$ *is generated by*

$$F = t_{r_6}t_{r_5}t_{r_4}t_{r_3}t_{r_2}t_{r_1}t_{r_5}t_{r_4}t_{r_8}, \quad G = t_{q_1}t_{q_2}t_{q_3}t_{q'_1}t_{q'_2}t_{q'_3}t_{q_0}$$

and its defining relations are $F^7 = G^3 = (GF)^2 = (GFG^{-1}F^{-1})^4 = 1$*. In the above presentation* r_i *are simple closed curves shown in Figure 8, and* q_i*,* q'_i *are simple closed curves shown in Figure 9.*

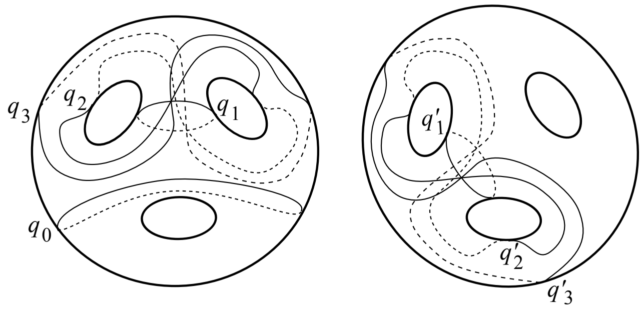

FIGURE 9

3.2. **A Dehn twist presentation of the action of $S_3 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4)$.** The finite group action on Σ_3 with the second largest order is the automorphism group of the Fermat quartic $\{(x : y : z) \in \mathbb{C}P^2 \,|\, x^4+y^4+z^4 = 0\}$. This group is isomorphic to $S_3 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4)$, where $S_3 = \langle x, y \mid x^2 = y^3 = 1, xyx^{-1} = y^{-1}\rangle$ acts on $\mathbb{Z}_4 \times \mathbb{Z}_4 = \langle z, w \mid z^4 = w^4 = 1, zw = wz\rangle$ by $xzx^{-1} = w, xwx^{-1} = z$, $yzy^{-1} = w$, $ywy^{-1} = (zw)^{-1}$.

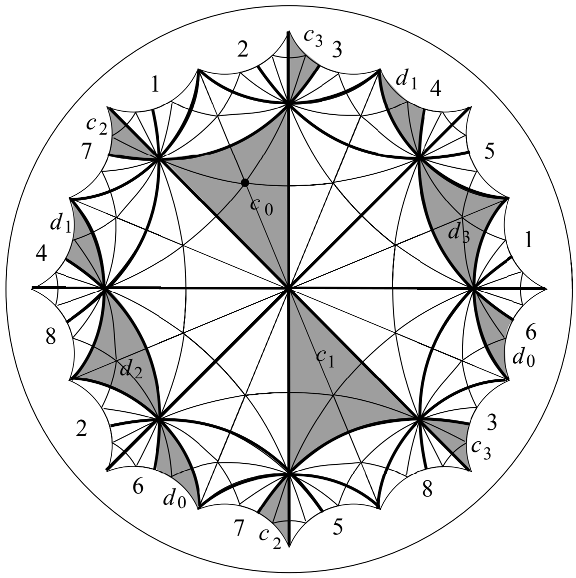

FIGURE 10

Figure 10 is obtained by editing Figure 10 of [7]. A fundamental region of the automorphism group of the Fermat quartic is a union of adjacent two triangles. Each thick triangles is a union of three fundamental regions. Figure 11 is obtained from Figure 10 by identifying each pair of edges with the same number. The triangles in Figure 11 correspond to the thick triangles in Figure 10. We obtained Figure 11 in the same way as in the previous subsection. Let P be a clockwise $2\pi/3$ rotation about the center of c_0 and Q a clockwise $2\pi/8$ rotation about the center of Figure 10. Then the

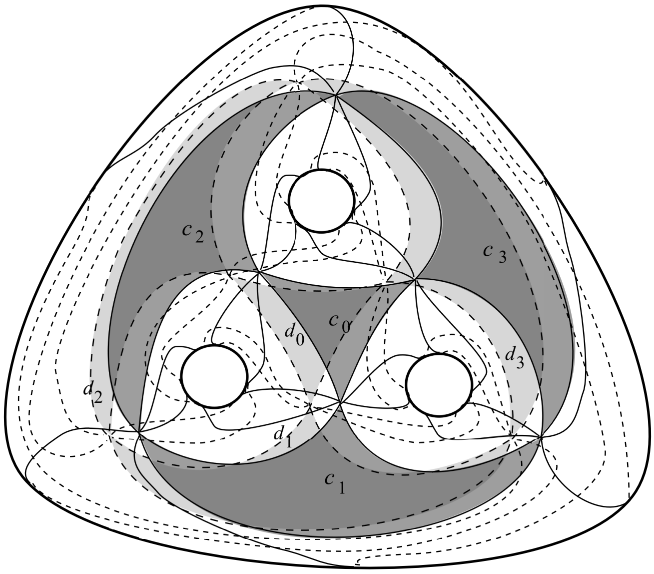

FIGURE 11

automorphism group of the Fermat quartic is generated by P and Q. On the left of

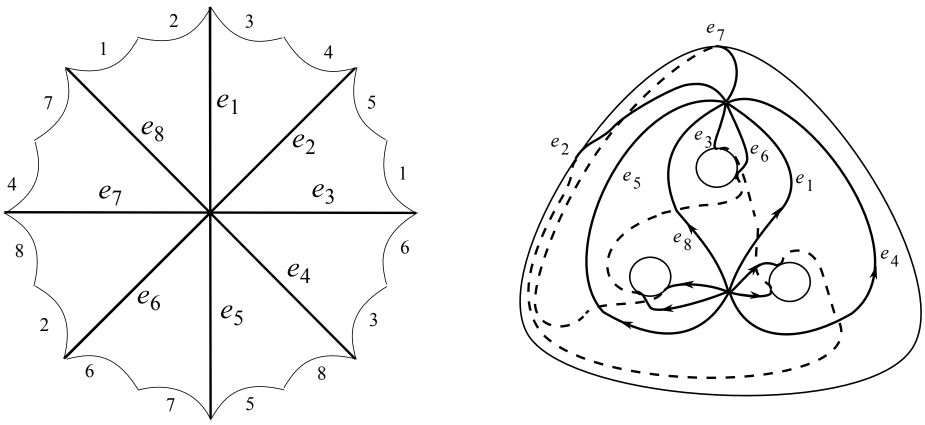

FIGURE 12

Figure 12, the rotation Q maps $e_1 \to e_2 \to \cdots \to e_8 \to e_1$. If we draw $e_1, \dots, e_8$ on Figure 11, we obtain the right of Figure 12. The total valency of Q is $\left(8, \frac{1}{8} + \frac{1}{4} + \frac{5}{8}\right)$, which corresponds to $f_{3,7}$ of Theorem 2. On upper left of Figure 13 excerpted from Figure 20 of [5], a product of Dehn twists $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 5 \cdot 4 \cdot 3 \cdot 8$ representing $f_{3,7}$ brings e_i to e_{i+1} $(i = 1, \dots, 6)$ and e_7 to e_1. Let Φ be an orientation preserving homeomorphism from the upper left of Figure 13 to the lower left of Figure 13 which sends e_i to e_i. The circles $s_i = \Phi(c_i)$ are as on the lower right of Figure 13 and we have $Q = t_{s_6} t_{s_5} t_{s_4} t_{s_3} t_{s_2} t_{s_5} t_{s_4} t_{s_3} t_{s_8}$.

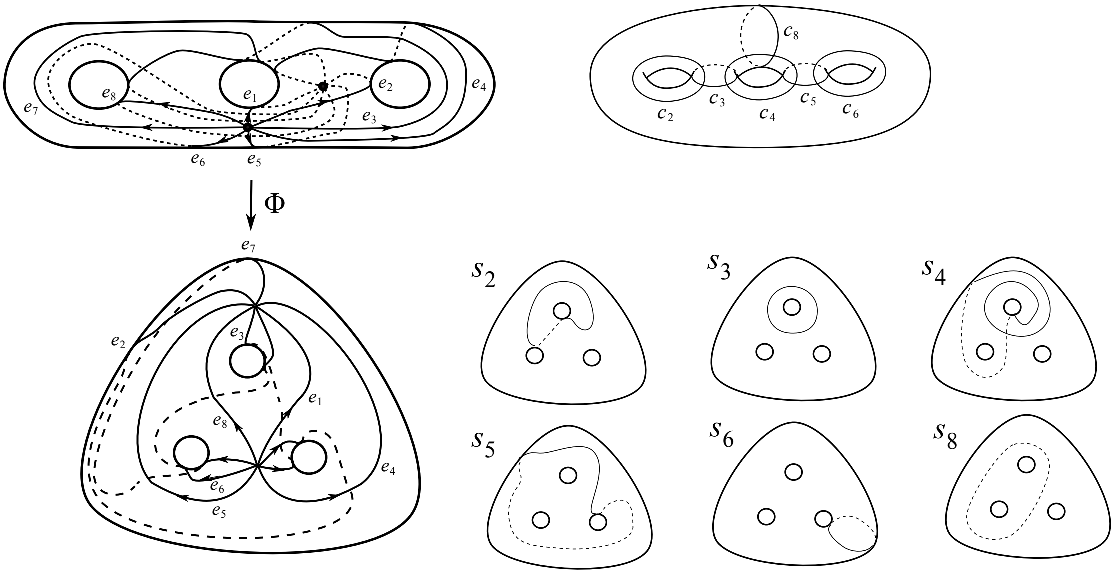

Figure 13

Proposition 6. *The automorphism group of the Fermat quartic* $\{(x : y : z) \in \mathbb{C}P^2 \,|\, x^4 + y^4 + z^4 = 0\}$ *is generated by*

$$P = t_{q_1} t_{q_2} t_{q_3} t_{q_1'} t_{q_2'} t_{q_3'} t_{q_0}, \quad Q = t_{s_6} t_{s_5} t_{s_4} t_{s_3} t_{s_2} t_{s_5} t_{s_4} t_{s_3} t_{s_8}$$

and its defining relations are $P^3 = Q^8 = (PQ)^2 = (PQ^4)^3 = 1$. *In the above presentation,* s_i *are simple closed curves shown in Figure 13, and* q_i *and* q_i' *are those in Figure 14.*

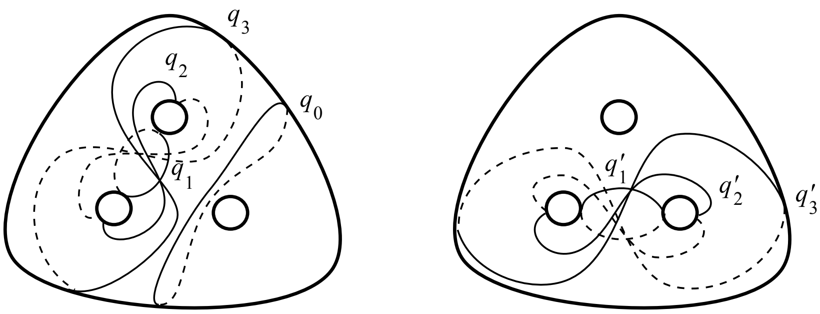

Figure 14

3.3. **A Dehn twist presentation of the action of** $\mathbb{Z}_2 \ltimes SL_2(3)$**.** This semi-direct product $\mathbb{Z}_2 \ltimes SL_2(3)$ is defined by the action of $\mathbb{Z}_2 = \langle \sigma \,|\, \sigma^2 = 1\rangle$ on $SL_2(3) = \langle \sigma_1, \sigma_2 | \sigma_1^3 = 1, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\rangle$ by $\sigma\sigma_1\sigma = \sigma_2$. If we put $T = \sigma_1$, $S = \sigma$, then $\mathbb{Z}_2 \ltimes SL_2(3) = \langle T, S \,|\, T^3 = S^2 = 1, (TS)^3 = (ST)^3\rangle$. In this group TS has order 12 and corresponds to $f_{3,6}$ of Theorem 2. Figure 15 obtained by modifying Figure 24 of [5] shows the action on Σ_3 of the product of Dehn twists $t_{c_6} t_{c_5} t_{c_4} t_{c_3} t_{c_2} t_{c_8}$ representing

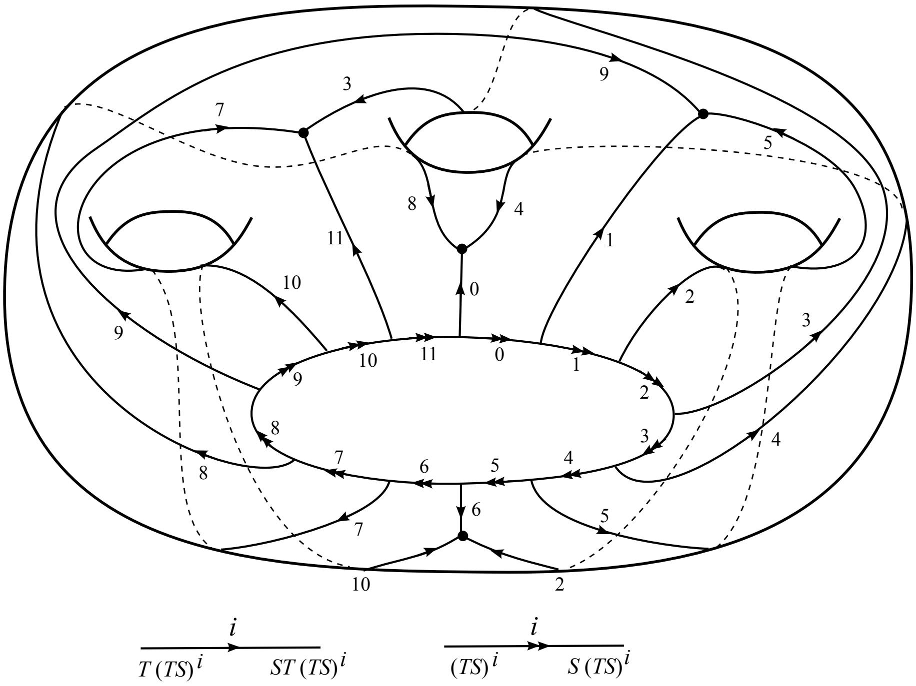

FIGURE 15

$f_{3,6}$. Let e_i be an edge with single arrow and index i, E_i be an edge with double arrow and index i, and $\bar{e}_i$, $\bar{E}_i$ be the edges with opposite orientation of e_i, E_i respectively. The group $\mathbb{Z}_2 \ltimes SL_2(3)$ acts on a graph in Σ_3 consists of these edges. We regard the left end F of the edge E_0 as the fundamental domain of this action and, for an elements g of $\mathbb{Z}_2 \ltimes SL_2(3)$, the end of an edge marked by the symbol g is Fg. The action of the involution S on these edges is as follows, $E_{11} \to \bar{e}_1$, $E_0 \to \bar{E}_0$, $E_1 \to e_0$, $E_2 \to \bar{e}_4$, $E_3 \to \bar{E}_3$, $E_4 \to e_3$, $E_5 \to \bar{e}_7$, $E_6 \to \bar{E}_6$, $E_7 \to e_6$, $E_8 \to \bar{e}_{10}$, $E_9 \to \bar{E}_9$, $E_{10} \to e_9$, $e_2 \to \bar{e}_8$, $e_5 \to \bar{e}_{11}$. For the circles $\gamma_1, \ldots, \gamma_6$, δ and ϵ on the left of Figure 16,

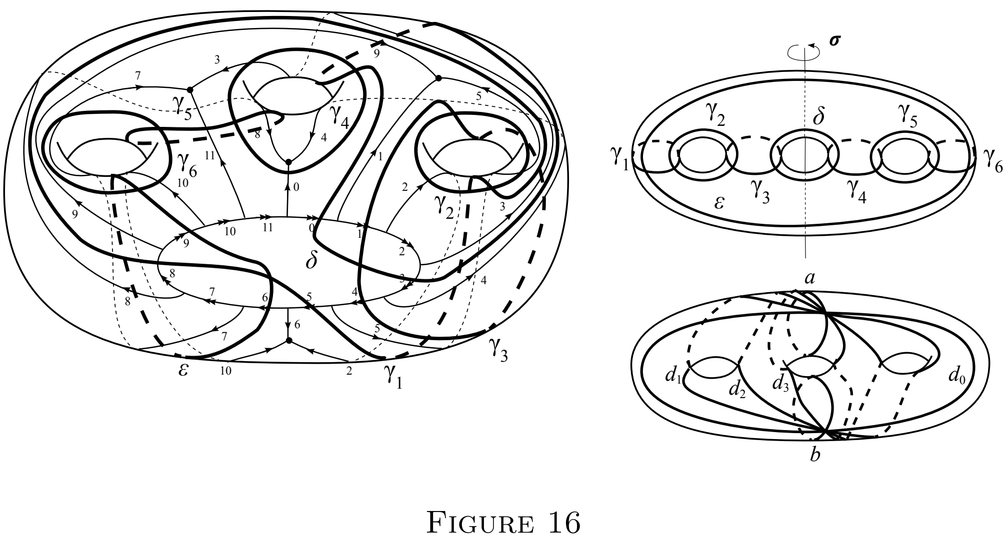

FIGURE 16

by investigating their intersection with edges e_i and E_i and the action of S on these edges, we see that S sends γ_1, γ_2, γ_3 to γ_6, γ_5, γ_4 respectively, and fixes δ and ϵ setwisely with reversing their orientations. There is an orientation preserving homeomorphism Φ between Σ_3 on the upper right of Figure 16 to Σ_3 on the left of Figure 16 sending the circles $\gamma_1, \dots, \gamma_6$, δ and ϵ to the circles with the same names. Korkmaz [10] showed that $\sigma = t_a^2 t_b^2 t_{d_3} t_{d_2} t_{d_1} t_{d_0}$, where Dehn twists are about the circles on the lower right of Figure 16. These circles d_1, d_2, d_3 are obtained from ϵ by the maps $t_{\gamma_1} t_{\gamma_6}$, $t_{\gamma_1} t_{\gamma_6} t_{\gamma_2} t_{\gamma_5}$, $t_{\gamma_1} t_{\gamma_6} t_{\gamma_2} t_{\gamma_5} t_{\gamma_3} t_{\gamma_4}$ respectively. We denote the images of the circles $a, b, d_0 = \epsilon, d_1, d_2, d_3$ by the map Φ by the same symbols and show them in Figure 17. The map $t_a^{-1} t_b^{-1}$ sends the circles d_1, d_2, d_3 on Σ_3 on the lower left of Figure 17 to the circles d_1', d_2', d_3' on the lower right of Figure 17. We remark that $t_a^2 t_b^2 t_{d_3} t_{d_2} t_{d_1} t_{d_0} = t_a t_b t_{d_3'} t_{d_2'} t_{d_1'} t_a t_b t_{d_0}$. In summary, we have:

Proposition 7. *The action of* $\mathbb{Z}_2 \ltimes SL_2(3) = \langle T, S \,|\, T^3 = S^2 = 1, (TS)^3 = (ST)^3 \rangle$ *on* Σ_3 *is generated by* $TS = t_{c_6} t_{c_5} t_{c_4} t_{c_3} t_{c_2} t_{c_8}$, $S = t_a t_b t_{d_3'} t_{d_2'} t_{d_1'} t_a t_b t_{d_0}$. *In the above presentation,* a, b, c_i, d_0, d_i' *are simple closed curves shown in Figure 17 .*

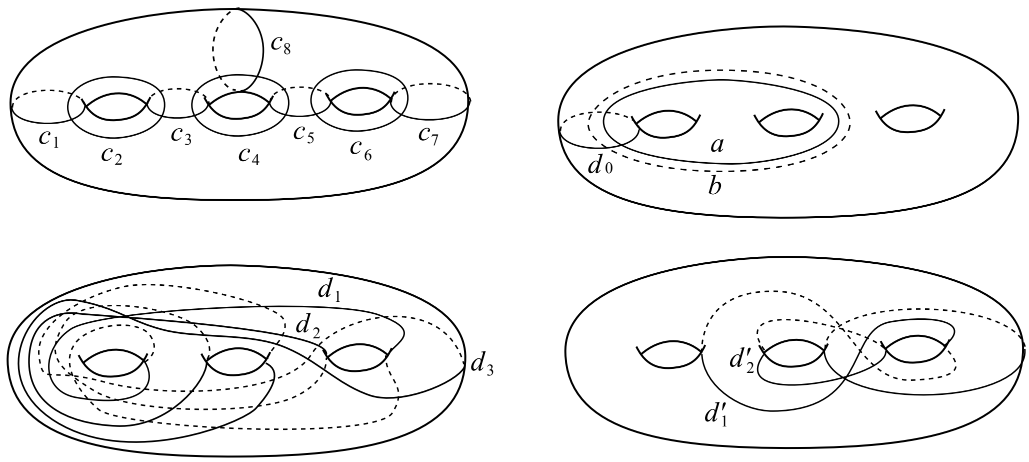

Figure 17

3.4. **Dehn twist presentations of the actions of** $D_{2,12,5}$**,** $\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_8)$**, and** $\mathbb{Z}_2 \times S_4$**.** These groups are presented as follows,

$$D_{2,12,5} = \langle x, y \,|\, x^2, y^{12}, xyxy^{-5} \rangle$$
$$\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_8) = \langle x, y, z \,|\, x^2, y^2, z^8, yzy^{-1}z^{-1}, xyx^{-1}y^{-1}, xzx^{-1}z^{-3}y^{-1} \rangle$$
$$\mathbb{Z}_2 \times S_4 = \langle x, y, z \,|\, x^2, y^3, z^4, xyx^{-1}y^{-1}, xzx^{-1}z^{-1}, yzyz \rangle.$$

These groups are subgroup of the hyperelliptic mapping class group of Σ_3. Yusuke Hasegawa [4] obtained Dehn twist presentations of the action of these groups. For self-containedness, we will explain Dehn twist presentations of these actions, which are obtained independently from the Hasegawa's presentations. The hyperelliptic mapping class group of Σ_3 is generated by Dehn twists about the circles $c_1, c_2, \ldots, c_7$. In our presentations, we use Dehn twists about these circles. In the following presentation, k means the Dehn twist t_{c_k} about the simple closed curve c_k and $\overline{k}$ means $t_{c_k}^{-1}$.

3.4.1. *A Dehn twist presentation of the action of $D_{2,12,5}$.* The group $D_{2,12,5}$ preserves a graph on Σ_3 illustrated in Figure 18. The edge with 1 is the fundamental domain F of this action and the edge with $g \in D_{2,12,5}$ is Fg. We denote the curve with edge xy^i by a_i $(i = 0, 1, \ldots, 5)$ and the curve with opposite orientation by $\bar{a}_i$. The element $x \in D_{2,12,5}$ maps $a_0 \to \bar{a}_0$, $a_1 \to \bar{a}_5$, $a_2 \to a_4$, $a_3 \to \bar{a}_3$, and the element $y \in D_{2,12,5}$ maps $a_0 \to a_1 \to a_2 \to a_3 \to a_4 \to a_5 \to \bar{a}_0 \to \bar{a}_1 \to \cdots$. By investigating the actions of Dehn twists on these circles, we see:

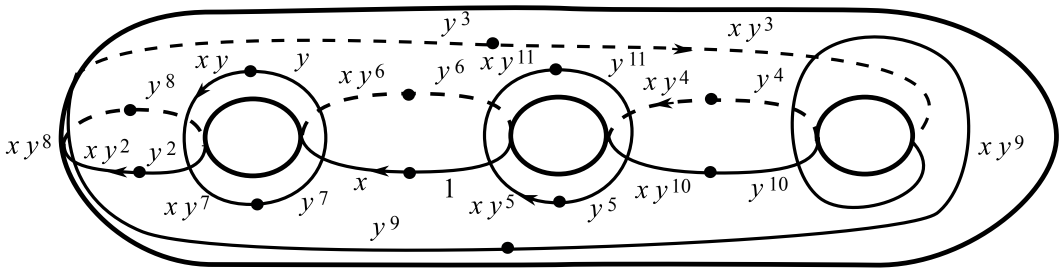

FIGURE 18

Proposition 8. *The action of $D_{2,12,5}$ on Σ_3 is generated by* $x = (1\cdot2\cdot3\cdot4\cdot5\cdot6)\cdot(1\cdot2\cdot3\cdot4\cdot5)\cdot(1\cdot2\cdot3\cdot4)\cdot(1\cdot2\cdot3)\cdot(1\cdot2)\cdot1\cdot(1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7)$, *and* $y = 1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot6$.

3.4.2. *A Dehn twist presentation of the action of $\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_8)$.* The group $\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_8)$ preserves a graph on Σ_3 illustrated in Figure 19. The edge with 1 is the fundamental domain F of this action and the edge with $g \in \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_8)$ is Fg. We denote the curve with edge z^i by b_i $(i = 0, 1, \ldots, 7)$ and the curve with opposite orientation by $\bar{b}_i$. The element $x \in \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_8)$ maps $b_0 \to \bar{b}_0$, $b_1 \to b_7$, $b_2 \to b_6$, $b_3 \to b_5$, $b_4 \to \bar{b}_4$, and the element $z \in \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_8)$ maps $b_0 \to b_1 \to b_2 \to b_3 \to b_4 \to b_5 \to b_6 \to b_7 \to b_0$. By investigating the actions of Dehn twists on these circles, we see:

Proposition 9. *The action of $\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_8)$ on Σ_3 is generated by* $x = (1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7)\cdot(1\cdot2\cdot3\cdot4\cdot5\cdot6)\cdot(1\cdot2\cdot3\cdot4\cdot5)\cdot(1\cdot2\cdot3\cdot4)\cdot(1\cdot2\cdot3)\cdot(1\cdot2)\cdot1$, *and* $z = 7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1$.

3.4.3. *A Dehn twist presentation of the action of $\mathbb{Z}_2 \times S_4$.* The action of the group S_4 preserves a graph on Σ_3 illustrated in Figure 20. The edge with 1234 is the fundamental

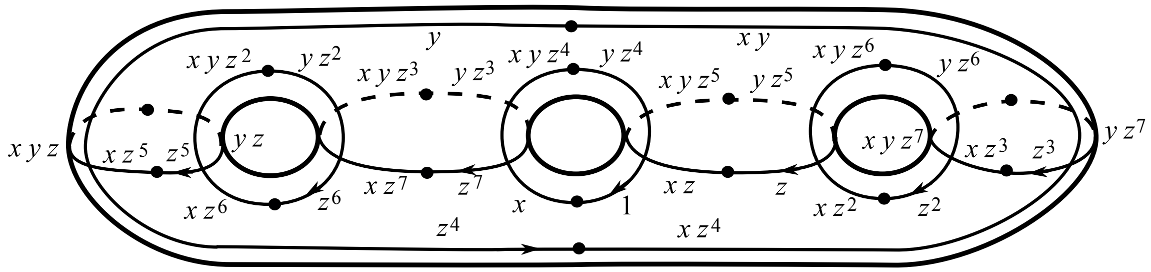

FIGURE 19

domain F of this action and the edge with $a_1a_2a_3a_4$ is $F\sigma$ for $\sigma \in S_4$ such that $\sigma(i) = a_i$ for each $i \in \{1,2,3,4\}$. We denote by d_i the circle having an arrow with the symbol d_i and by $\bar{d}_i$ the circle with opposite orientation. The element $x \in \mathbb{Z}_2 \times S_4$ acts on Σ_3 as a hyperelliptic involution. The cyclic permutation $y = (2,3,4) \in \mathbb{Z}_2 \times S_4$ maps $d_1 \to d_5 \to d_4 \to d_1$, $d_2 \to d_3 \to \bar{d}_6 \to d_2$, and the cyclic permutation $z = (1,4,3,2) \in \mathbb{Z}_2 \times S_4$ maps $d_1 \to d_2 \to \bar{d}_1$, $d_3 \to d_4 \to \bar{d}_3$, $d_5 \to d_6 \to d_5$. By investigating the actions of Dehn twists on these circles, we see:

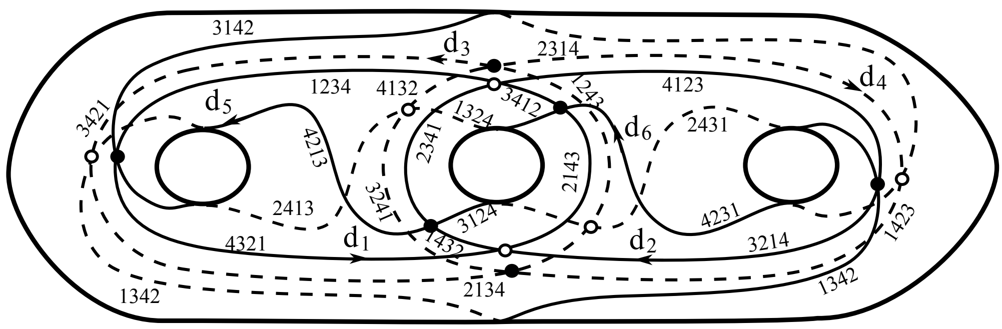

FIGURE 20

Proposition 10. *The action of* $\mathbb{Z}_2 \times S_4$ *on* Σ_3 *is generated by* $x = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1$, $y = 5\cdot 6\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 2\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1$ *and* $z = (1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7)^2$.

4. A LIST OF NON MAXIMAL FINITE GROUP ACTIONS ON Σ_3

In this section, we list non maximal finite group actions on Σ_3 as subgroup actions of maximal finite group actions. This list is obtained by using GAP 4. In this list, 3.xx is a name of a finite group action on the list by Broughton [2], especially, 3.at = $PSL_2(7)$ (§2.1), 3.as = $S_3 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4)$ (§2.2), 3.ao = $\mathbb{Z}_2 \ltimes SL_2(3)$(§2.3), 3.ap = $\mathbb{Z}_2 \times S_4$, 3.am.1 = $\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_8)$, 3.ah = $D_{2,12,5}$ (§2.4), 3.aa = $\mathbb{Z}_{14}$, 3.t = $\mathbb{Z}_9$ (§1).

$$\underline{\text{3.xx}}:\ \text{3.yy} \ni F_1 = ***, F_2 = ***, [\cdots]$$

means that 3.xx is a subgroup action of 3.yy, the elements F_1, F_2 of 3.yy generate the action of 3.xx, and, in $[\cdots]$, $\cdots$ are defining relations among F_1, F_2.

3.aq : 3.as $\ni F_1 = P, F_2 = Q^{-2}$, $[F_1^3, F_2^4, F_2^{-1}F_1^{-1}F_2^{-1}F_1^{-1}F_2^{-1}F_1^{-1}, F_2^{-1}F_1F_2^{-1}F_1F_2^{-1}F_1]$

3.am.2 : 3.as $\ni F_1 = Q, F_2 = PQ^{-1}P$, $[F_2^2, F_1F_2F_1^{-2}F_2F_1, F_1^8, F_2F_1F_2F_1F_2F_1F_2F_1]$

3.al : 3.at $\ni F_1 = F^{-3}GF^{-2}, F_2 = FG^2F$, $[F_1^3, F_2F_1F_2F_1, F_2^4, F_2^2F_1^{-1}F_2F_1^{-1}F_2^{-2}F_1^{-1}F_2F_1^{-1}]$

3.as $\ni F_1 = QP, F_2 = PQ^2, F_3 = PQ^{-1}P$,

$[F_1^2, F_3^2, F_2^3, F_1F_2F_3F_2^{-1}, F_3F_1F_2^{-1}F_3F_1F_2^{-1}, F_3F_1F_3F_1F_3F_1]$

3.ap $\ni F_1 = yx^{-1}, F_2 = z$, $[F_1^2, F_2^3, F_2F_1F_2F_1F_2F_1F_2F_1]$

3.ak : 3.ap $\ni F_1 = y, F_2 = z$, $[F_1^2, F_2^3, F_2F_1F_2F_1F_2F_1F_2F_1]$

3.aj : 3.ao $\ni F_1 = T, F_2 = STS^{-1}$, $[F_1^3, F_2^3, F_2F_1F_2F_1^{-1}F_2^{-1}F_1^{-1}]$

3.ai : 3.ap $\ni F_1 = x, F_2 = z, F_3 = yzy^{-1}$, $[F_1^2, F_2^3, F_3^3, F_2F_1F_2^{-1}F_1, F_3F_1F_3^{-1}F_1, F_3F_2F_3F_2]$

3.ag †: 3.at $\ni F_1 = F, F_2 = GF^{-3}G^{-1}FG^{-1}$, $[F_2^3, F_1F_2F_1^{-2}F_2^{-1}, F_1F_2F_1F_2F_1F_2, F_1^7]$

3.ad.1 : 3.ap $\ni F_1 = x, F_2 = y, F_3 = zyz$, $[F_1^2, F_2^2, F_2F_1F_2F_1, F_3F_1F_3^{-1}F_1, F_3F_2F_3F_2, F_3^4]$

3.am.1 $\ni F_1 = x, F_2 = y, F_3 = z^{-2}$, $[F_1^2, F_2^2, F_1F_3^{-1}F_1F_3^{-1}, F_2F_1F_2F_1, F_3F_2F_3^{-1}F_2, F_3^4]$

3.ad.2 : 3.ao $\ni F_1 = S, F_2 = TST^{-1}, F_3 = T^{-1}ST$, $[F_1^2, F_2^2, F_3^2, F_1F_3F_2F_1F_2F_3, F_2F_1F_3F_1F_2F_3]$

3.ac.1 : 3.as $\ni F_1 = Q^{-2}, F_2 = PQ^{-2}P^{-1}$, $[F_2^{-1}F_1^{-1}F_2F_1, F_1^4, F_2^4]$

3.ac.2 : 3.am.1 $\ni F_1 = zx^{-1}, F_2 = z^{-1}x^{-1}$, $[F_2^4, F_2^2F_1^2, F_2^{-1}F_1^{-1}F_2^{-1}F_1F_2^{-1}F_1^{-1}F_2^{-1}F_1]$

3.ab.1 : 3.am.1 $\ni F_1 = y, F_2 = z$, $[F_1^2, F_2F_1F_2^{-1}F_1, F_2^8]$

3.ab.2 : 3.as $\ni F_1 = Q^{-1}P, F_2 = Q^3P$, $[F_2F_1^{-1}F_2F_1^{-1}, F_2^2F_1^2, F_2^8, F_1^8]$

3.z : 3.at $\ni F_1 = G^{-1}F^{-1}, F_2 = F^{-3}GF^{-2}$, $[F_1^2, F_2^3, F_1F_2^{-1}F_1F_2^{-1}F_1F_2^{-1}]$

3.as $\ni F_1 = PQ^2, F_2 = PQ^{-4}P^{-1}Q^{-2}P^{-1}$, $[F_2^3, F_1^3, F_2F_1F_2F_1, F_2F_1^{-1}F_2F_1^{-1}F_2F_1^{-1}]$

3.ap $\ni F_1 = z, F_2 = yzy^{-1}$, $[F_1^3, F_2^3, F_2F_1F_2F_1]$

3.y : 3.ap $\ni F_1 = x, F_2 = z, F_3 = yzyz^{-1}y^{-1}$, $[F_1^2, F_3^2, F_2^3, F_2F_3F_2F_3, F_2F_1F_2^{-1}F_1, F_3F_1F_3F_1]$

3.ah $\ni F_1 = y^{-2}, F_2 = y^2x^{-1}y^{-6}$, $[F_2^2, F_2F_1F_2F_1, F_1^6]$

3.x : 3.ah $\ni F_1 = yx^{-1}, F_2 = y^{-1}x^{-1}$, $[F_1^{-2}F_2^2, F_1^{-2}F_2^{-2}, F_2^{-1}F_1^{-1}F_2F_1F_2F_1]$

3.v : 3.ao $\ni F_1 = T, F_2 = STSTS^{-1}$, $[F_1^3, F_2F_1F_2^{-1}F_1^{-1}, F_1F_2^4]$

3.u : 3.ah $\ni F_1 = y$, $[F_1^{12}]$

3.s.1 : 3.as $\ni F_1 = Q^{-1}PQ^{-2}, F_2 = Q^{-3}P$, $[F_2F_1^{-1}F_2^{-1}F_1^{-1}, F_2^4, F_2^2F_1^2, F_2^2F_1^{-2}]$

3.so $\ni F_1 = STS^{-1}T^{-1}, F_2 = ST^{-1}S^{-1}T$, $[F_2F_1F_2F_1^{-1}, F_1^{-1}F_2F_1^{-1}F_2^{-1}]$

3.s.2 : 3.am.1 $\ni F_1 = z^{-2}y^{-1}, F_2 = z^{-2}y^{-1}xz^2$, $[F_2^2, F_1^4, F_1F_2F_1F_2]$

3.ap $\ni F_1 = y, F_2 = zyzx^{-1}$, $[F_1^2, F_2F_1F_2F_1, F_2^4]$

3.r.1 : 3.ap $\ni F_1 = x, F_2 = y, F_3 = zyz^{-1}y^{-1}z$,

$[F_1^2, F_2^2, F_3^2, F_2F_3F_2F_3, F_2F_1F_2F_1, F_3F_1F_3F_1, F_2F_3F_1F_2F_3F_1]$

3.am.1 $\ni F_1 = x, F_2 = y, F_3 = y^{-1}zxzx$,

$[F_1^2, F_2^2, F_3^2, F_1F_3F_1F_3, F_3F_2F_3F_2, F_2F_1F_2F_1, F_2F_3F_1F_2F_3F_1]$

†its branching indices is $(3, 3, 7)$

3.r.2 : 3.at $\ni F_1 = F^{-1}GF^{-1}GF, F_2 = GF^{-2}GF^{-2}$, $[F_1^2, F_2^2, F_1F_2F_1F_2F_1F_2F_1F_2]$
3.as $\ni F_1 = PQ^{-1}P, F_2 = PQ^3P$, $[F_1^2, F_2^4, F_1F_2^{-1}F_1F_2^{-1}]$
3.ao $\ni F_1 = TST^{-1}, F_2 = T^{-1}ST,$, $[F_1^2, F_2^2, F_1F_2F_1F_2F_1F_2F_1F_2]$
3.am.1 $\ni F_1 = x, F_2 = yz^2$, $[F_1^2, F_2^{-1}F_1F_2^{-1}F_1, F_2^4]$
3.q.1 (x, x, y^{-1}, y): 3.ap $\ni F_1 = x, F_2 = z^{-1}y^{-1}$, $[F_1^2, F_2F_1F_2^{-1}F_1, F_2^4]$
3.am.1 $\ni F_1 = y, F_2 = z^{-2}$, $[F_1^2, F_2^{-1}F_1F_2F_1, F_2^4]$
3.q.1 (x, xy^2, y, y): 3.as $\ni F_1 = Q^{-2}, F_2 = PQ^{-1}P$, $[F_2^2, F_1^4, F_2F_1F_2F_1^{-1}]$
3.ao $\ni F_1 = S, F_2 = TSTS^{-1}T$, $[F_1^2, F_2^4, F_1F_2F_1F_2^-1]$
3.q.1 (x, y^2, xy, y): 3.ah $\ni F_1 = yxy, F_2 = y^3$, $[F_1^2, F_2F_1F_2^{-1}F_1, F_2^4]$
3.am.1 $\ni F_1 = y, F_2 = y^{-1}xz$, $[F_1^2, F_2F_1F_2^{-1}F_1, F_2^4]$
3.q.2 : 3.ap $\ni F_1 = z^{-1}y^{-1}, F_2 = zyz^{-1}x^{-1}$, $[F_2^2, F_1^4, F_2F_1F_2F_1]$
3.am.1 $\ni F_1 = x, F_2 = z^{-2}$, $[F_1^2, F_2F_1F_2F_1, F_2^4]$
3.p (x^6, x, x): 3.am.1 $\ni F_1 = z$, $[F_1^8]$ 3.p (x^2, x, x^5): 3.as $\ni F_1 = Q^{-1}P$, $[F_1^8]$
3.o (x, x, x^5): 3.aa $\ni F_1 = x^6$, $[F_1^7]$ 3.o (x, x^2, x^4): 3.at $\ni F_1 = F$, $[F_1^7]$
3.n : 3.ap $\ni F_1 = z, F_2 = yzyz^{-1}y^{-1}$, $[F_2^2, F_1^3, F_1F_2F_1F_2]$
3.ah $\ni F_1 = xy^{-2}, F_2 = xy^{-6}$, $[F_2^2, F_1^2, F_2F_1F_2F_1F_2F_1]$
3.m : 3.at $\ni F_1 = G^{-1}F^{-1}, F_2 = G^{-1}FG^{-1}F^3G^{-1}F$, $[F_1^2, F_2^3, F_1F_2F_1F_2]$
3.as $\ni F_1 = QP, F_2 = Q^{-1}PQ^3P^{-1}$, $[F_1^2, F_2^3, F_1F_2F_1F_2]$
3.ap $\ni F_1 = z, F_2 = yzyz^{-1}y^{-1}x^{-1}$, $[F_2^2, F_1^3, F_2F_1^{-1}F_2F_1^{-1}]$
3.ah $\ni F_1 = x, F_2 = yx^{-1}y^{-1}x$, $[F_1^2, F_2^3, F_1F_2^{-1}F_1F_2^{-1}]$
3.k : 3.ao $\ni F_1 = STST^{-1}S^{-1}TS^{-1}$, $[F_1^6]$
3.j : 3.ap $\ni F_1 = x, F_2 = z$, $[F_1^2, F_2^3, F_2F_1F_2^{-1}F_1]$; 3.ah $\ni F_1 = y^{-2}$, $[F_1^6]$
3.i.1 : 3.at $\ni F_1 = F^{-2}GF^{-1}GF^{-3}$, $[F_1^4]$; 3.as $\ni F_1 = Q^{-3}P$, $[F_1^4]$
3.ao $\ni F_1 = TSTS^{-1}T$, $[F_1^4]$; 3.am.1 $\ni F_1 = yz^2$, $[F_1^4]$; 3.ap $\ni F_1 = xyz$, $[F_1^4]$
3.i.2 : 3.am.1 $\ni F_1 = x, F_2 = y$, $[F_1^2, F_2^2, F_1F_2F_1F_2]$
3.ap $\ni F_1 = yx^{-1}, F_2 = zyz^{-1}y^{-1}zx^{-1}$, $[F_1^2, F_2^2, F_1F_2F_1F_2]$
3.h (x, x, y, y, xy, xy): 3.at $\ni F_1 = G^{-1}F^{-1}, F_2 = GF^{-2}GF^{-2}$, $[F_1^2, F_2^2, F_2F_1F_2F_1]$
3.as $\ni F_1 = PQ^{-1}P, F_2 = Q^{-4}$, $[F_1^2, F_2^2, F_1F_2F_1F_2]$
3.ao $\ni F_1 = S, F_2 = TST^{-1}STS^{-1}T^{-1}$, $[F_1^2, F_2^2, F_2F_1F_2F_1]$
3.am.1 $\ni F_1 = x, F_2 = zxzy^{-1}$, $[F_1^2, F_2^2, F_2F_1F_2F_1]$
3.ap $\ni F_1 = zyz^{-1}x^{-1}, F_2 = z^{-1}yz^{-1}y^{-1}$, $[F_1^2, F_2^2, F_1F_2F_1F_2]$
3.h (x, x, y, y, y, y): 3.ap $\ni F_1 = x, F_2 = zyz^{-1}y^{-1}zy^{-1}$, $[F_1^2, F_2^2, F_2F_1F_2F_1]$
3.am.1 $\ni F_1 = x, F_2 = zxz$, $[F_1^2, F_2^2, F_1F_2F_1F_2]$
3.ah $\ni F_1 = x, F_2 = yxy$, $[F_1^2, F_2^2, F_2F_1F_2F_1]$
3.g : 3.am.1 $\ni F_1 = z^{-1}x^{-1}$, $[F_1^4]$; 3.ah $\ni F_1 = y^3x^{-1}$, $[F_1^4]$
3.f (x, x, x, x): 3.as $\ni F_1 = PQ^{-2}P^{-1}$, $[F_1^4]$; 3.ao $\ni F_1 = STSTS^{-1}T$, $[F_1^4]$
3.f (x, x, x^-1, x^-1): 3.ap $\ni F_1 = z^{-1}y^{-1}$, $[F_1^4]$; 3.am.1 $\ni F_1 = z^{-2}$, $[F_1^4]$

3.e : 3.at $\ni F_1 = GF^{-2}G^{-1}F^2G^{-1}F$, $[F_1^3]$; 3.as $\ni F_1 = Q^{-1}PQ^3P^{-1}$, $[F_1^3]$
3.ap $\ni F_1 = z$, $[F_1^3]$; 3.ah $\ni F_1 = y^4$, $[F_1^3]$
3.d : 3.ao $\ni F_1 = T$, $[F_1^3]$; 3.t $\ni F_1 = x^3$, $[F_1^3]$
3.c : 3.ap $\ni F_1 = zyz^{-1}y^{-1}zy^{-1}$, $[F_1^2]$; 3.am.1 $\ni F_1 = y$, $[F_1^2]$; 3.ah $\ni F_1 = yxy$, $[F_1^2]$
3.b : 3.at $\ni F_1 = G^{-1}F^{-1}$, $[F_1^2]$; 3.as $\ni F_1 = PQ^{-1}PQ^4$, $[F_1^2]$; 3.ao $\ni F_1 = S$, $[F_1^2]$
3.ap $\ni F_1 = xyzyz^{-1}y^{-1}z$, $[F_1^2]$; 3.am.1 $\ni F_1 = x$, $[F_1^2]$; 3.ah $\ni F_1 = x$, $[F_1^2]$
3.a : 3.ap $\ni F_1 = x$, $[F_1^2]$; 3.am.1 $\ni F_1 = zxzx^{-1}$, $[F_1^2]$; 3.ah $\ni F_1 = xyxy$, $[F_1^2]$; 3.aa $\ni F_1 = x^7$, $[F_1^2]$
an action which is not in the list by Broughton an action of $\langle b, c \,|\, b^2 = c^4 = 1, bcb = c^{-1}\rangle$(order = 16)
3.as $\ni F_1 = Q^-2, F_2 = PQ^{-1}P, F_3 = PQ^3P,$
$[F_2^2, F_3^{-1}F_1F_3^{-1}F_1, F_3^4, F_3^{-2}F_1^2, F_2F_1^{-1}F_2F_1, F_3F_2F_3F_2]$

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