DEHN TWIST PRESENTATIONS OF FINITE GROUP ACTIONS
ON THE ORIENTED SURFACES OF GENUS 3

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Abstract. In this note we give presentations of all finite subgroups of the mapping class group of a closed surface of genus 3 by Dehn twists up to conjugacy.

1. Introduction

Let $\Sigma_g$ be a closed oriented surface of genus $g$, and $\text{Homeo}_+ (\Sigma_g)$ a group of orientation preserving homeomorphisms over $\Sigma_g$. In this paper, for two elements $f_1, f_2 \in \text{Homeo}_+ (\Sigma_g)$, $f_1 f_2$ means applying $f_1$ and then $f_2$. The group $\mathcal{M}(\Sigma_g)$ of isotopy classes of $\text{Homeo}_+ (\Sigma_g)$ is called the mapping class group of $\Sigma_g$. Any finite subgroup of $\mathcal{M}(\Sigma_g)$ can be considered as the automorphism group of some algebraic curve, and hence finite subgroups of $\mathcal{M}(\Sigma_g)$ are investigated in various contexts. On the other hand, Dehn [3] and Lickorish [11] proved that $\mathcal{M}(\Sigma_g)$ is generated by Dehn twists. For a simple closed curve $c$ on $\Sigma_g$, the homeomorphism $t_c$ on $\Sigma_g$ indicated in Figure 1 is called the Dehn twist about $c$. Since actions on Dehn twists on geometric objects on $\Sigma_g$, for example homology groups of $\Sigma_g$, are easy to understand, presentations of elements in $\mathcal{M}(\Sigma_g)$ by Dehn twists are useful for the investigation on $\mathcal{M}(\Sigma_g)$. For periodic elements in $\mathcal{M}(\Sigma_g)$, Ishizaka [6] completely obtained Dehn twist presentations in hyperelliptic case, and the author [5] obtained Dehn twist presentations when $g$ is at most 4. Nakamura-Nakanishi [12] obtained Dehn twist presentations of finite groups actions on $\Sigma_2$. In this note, we investigate on Dehn twist presentations of finite group actions on $\Sigma_3$.

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2. Dehn twist presentations for the periodic maps over \( \Sigma_3 \)

At first, we explain the classification of periodic map over \( \Sigma_3 \) by Nielsen [13]. A homeomorphism \( f \) over \( \Sigma_3 \) is called periodic if there is an integer \( n \geq 1 \) which satisfies \( f^n = id_{\Sigma_3} \). The least integer \( n \) which satisfies the above condition is called the period of \( f \). Let \( n \) be the period of periodic map \( f \) over \( \Sigma_3 \). A point \( p \) on \( \Sigma_3 \) is a multiple point if there is an integer \( k \) such that \( 0 < k < n \) and \( f^k(p) = p \), and \( M_f \subset \Sigma_3 \) denotes the set of multiple points of \( f \). Let \( \Sigma_3/f \) be the orbit space of \( f \), and \( \pi_f: \Sigma_3 \to \Sigma_3/f \) the quotient map, i.e., a map defined by \( \pi_f(x) = [x] \), where \([x]\) is the point on \( \Sigma_3/f \) represented by the point \( x \in \Sigma_3 \). Then \( \pi_f \) is an \( n \)-fold branched covering ramified at \( \pi_f(M_f) \subset \Sigma_3/f \). Hence, we call \( B_f = \pi_f(M_f) \) a set of branch point. In the above situation, \( \pi_f|_{\Sigma_3 \setminus M_f} : \Sigma_3 \setminus M_f \to (\Sigma_3/f) \setminus B_f \) is an \( n \)-fold covering in an ordinary meaning.

We define a homomorphism \( \Omega_f : \pi_1((\Sigma_3/f) \setminus B_f, x) \to \mathbb{Z}_n \) describing this \( n \)-fold covering \( \pi_f|_{\Sigma_3 \setminus M_f} \) as follows. We choose a point \( \hat{x} \) in \( \Sigma_3 \) such that \( \pi_f(\hat{x}) = x \). Let \( l : [0, 1] \to (\Sigma_3/f) \setminus B_f \) be a loop satisfying \( l(0) = l(1) = x \). We make a lift \( \hat{l} : [0, 1] \to \Sigma_3 \) of \( l \) which begins from \( \hat{l}(0) = \hat{x} \), then we see \( \pi_f(\hat{l}(1)) = l(1) = x \), i.e., \( \hat{l}(1) \) is in the orbit of \( \hat{l}(0) = \hat{x} \) by the periodic map \( f \), hence there is an integer \( k \) such that \( f^k(\hat{x}) = \hat{l}(1) \).

We set \( \Omega_f([l]) = k \in \mathbb{Z}_n \). Since \( \mathbb{Z}_n \) is an Abelian group, this homomorphism \( \Omega_f \) induces a homomorphism \( \omega_f : H_1((\Sigma_3/f) \setminus B_f) \to \mathbb{Z}_n \).

Two periodic maps \( f, f' \) over \( \Sigma_3 \) are conjugate if there is a homomorphism \( g \) from \( \Sigma_3 \) to itself such that \( f' = g \circ f \circ g^{-1} \).

For each branch point \( Q_i \in B_f \) of \( f \), we choose mutually disjoint small disk \( D_{Q_i} \) and orient the boundary \( S_{Q_i} \) of \( D_{Q_i} \) clockwisely.

**Theorem 1.** [13] Two periodic maps \( f, f' \) over \( \Sigma_3 \) are conjugate if and only if the following three conditions are satisfied,

1. the period of \( f = \) the period of \( f' \),
2. the number of elements of \( B_f = \) the number of elements of \( B_{f'} \),
3. if we change the numbering of \( B_f = \{Q'_1, Q'_2, \ldots, Q'_b\} \) properly, we have \( \omega_f(S_{Q_i}) = \omega_{f'}(S_{Q'_i}) \) for each \( i \).

By the above theorem, the notation \( \left( n, \frac{\theta_1}{n} + \cdots + \frac{\theta_b}{n} \right) \), where \( n = \) the period of \( f \), \( \theta_i = \omega_f(S_{Q_i}) \), completely determines the conjugacy class of \( f \). This notation is introduced by Ashikaga and Ishizaka [1] and called total valency.

Dehn twist presentations of periodic maps over \( \Sigma_3 \) are obtained as follows.

**Theorem 2.** [5] Any periodic map on \( \Sigma_3 \) is a power of some of the following periodic maps,
Up to conjugacy, these periodic maps are presented as products of Dehn twists as follows. In the following presentations $k$ means the Dehn twist about the simple closed curve $c_k$ in Figure 2.

\[
\begin{align*}
f_{3,1} &= \left(\frac{14}{14}, \frac{1}{7}, \frac{3}{2}, \frac{1}{2}\right), \quad f_{3,2} = \left(\frac{12}{12}, \frac{1}{12}, \frac{5}{12}, \frac{1}{2}\right), \quad f_{3,3} = \left(8, \frac{1}{8}, \frac{1}{8}, \frac{3}{4}\right), \\
f_{3,4} &= \left(4, \frac{1}{2}, \frac{1}{2}\right), \quad f_{3,5} = (2), \quad f_{3,6} = \left(\frac{12}{12}, \frac{1}{4}, \frac{2}{3}\right), \quad f_{3,7} = \left(8, \frac{1}{8}, \frac{1}{4}, \frac{5}{8}\right), \\
f_{3,8} &= \left(9, \frac{1}{9}, \frac{1}{3}, \frac{5}{9}\right), \quad f_{3,9} = \left(7, \frac{1}{7}, \frac{2}{7}, \frac{4}{7}\right).
\end{align*}
\]

3. Maximal finite group actions over $\Sigma_3$ and their Dehn twist presentations

An injection $\epsilon$ from a finite group $G$ to $\text{Homeo}_+\left(\Sigma_g\right)$ is called the action of $G$ over $\Sigma_g$. In this paper, the group $G$ acts on $\Sigma_3$ from the right; the action of $g \in G$ on $x \in \Sigma_3$ is written as $x\epsilon(g)$ or $xg$. For a system of generators $\{g_1, \ldots, g_k\}$ of $G$, we call a system of Dehn twist presentations of the isotopy classes of $\{\epsilon(g_1), \ldots, \epsilon(g_k)\}$ a Dehn twist presentation for the finite group action $\epsilon : G \to \text{Homeo}_+\left(\Sigma_g\right)$. Two finite group actions $\epsilon_1, \epsilon_2 : G \to \text{Homeo}_+\left(\Sigma_g\right)$ are equivalent if there is an automorphism $\omega$ of $G$ and an orientation preserving homeomorphism $h$ over $\Sigma_g$ which satisfy $\epsilon_2(g) = h^{-1}\epsilon_1(\omega(g))h$ for any $g \in G$.

For an action of a finite group $G$ over $\Sigma_g$, $\epsilon : G \to \text{Homeo}_+\left(\Sigma_g\right)$ and a subgroup $H$ of $G$, we can define an action of $H$ over $\Sigma_g$ by the restriction $\epsilon|_H : H \to \text{Homeo}_+\left(\Sigma_g\right)$ and we call this action a subgroup action of $\epsilon$. If we obtain a Dehn twist presentation of a
action $\epsilon$ of $G$ over $\Sigma_g$, then we obtain a Dehn twist presentation of a subgroup action $\epsilon|_{H}$ automatically. Therefore, we will obtain Dehn twist presentations of maximal finite group actions over $\Sigma_3$.

We remark here that we checked Dehn twist presentations by using T4M7 * implemented by K. Ahara, T. Sakasai, M. Suzuki.

Based on the classification of finite group actions on $\Sigma_3$ by Broughton [2], we see:

**Proposition 3.** Any finite group action on $\Sigma_3$ is a subgroup action of the actions of following groups:

$$\mathbb{Z}_9, \mathbb{Z}_{14}, D_{2,12,5}, \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_8), \mathbb{Z}_2 \times S_4, \mathbb{Z}_2 \ltimes SL_2(3), S_3 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4), PSL_2(7).$$

**Remark 4.** 1. In general, for a finite group $G$, its action on $\Sigma_g$ is not unique. Nevertheless, for each of the above eight groups, its action on $\Sigma_3$ is unique up to equivalence.
2. In §3, we make a list of other finite group actions as subgroup actions of the above eight finite group actions.
3. On Broughton’s list, there is no action of $\mathbb{Z}_2 \ltimes SL_2(3)$.
4. The actions of $\mathbb{Z}_9$, $\mathbb{Z}_{14}$ are generated by $f_{3,8}$ and $f_{3,1}$ in Theorem 2 respectively and Dehn twist presentations of them are obtained in this theorem.

3.1. **A Dehn twist presentation of the action of $PSL_2(7)$**. By Hurwitz, it was shown that orders of finite subgroups of $\mathcal{M}(\Sigma_g)$ are at most $84(g-1)$. When $g = 3$ there is a subgroup of $\mathcal{M}(\Sigma_3)$ whose order is $84(3 - 1) = 168$. This subgroup is the

* Downloadable from http://www.ms.u-tokyo.ac.jp/~sakasai/MCG/MCG.html
automorphism group of the Klein quartic \( \{(x : y : z) \in \mathbb{C}P^2 \mid x^3 y + y^3 z + z^3 y = 0\} \),
and is isomorphic to
\[
PSL_2(7) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}_7, \ ad - bc = 1 \right\} / \{ \pm E_2 \}.
\]

Figure 3 explains the action of \( PSL_2(7) \) on \( \Sigma_3 \). Let \( G \) be the clockwise \( 2\pi/3 \) rotation whose center is \( p_1 \) and \( F \) the clockwise \( 2\pi/7 \) rotation whose center is \( p_2 \). Then the action of \( PSL_2(7) \) is generated by \( F \) and \( G \), and the relations are \( F^7 = G^3 = (GF)^2 = (GFG^{-1}F^{-1})^4 = 1 \). Figure 4 is obtained from Figure 3 by identifying each pair of edges with the same number. The triangles in Figure 4 correspond to the thick triangles in Figure 3; for example, the triangle with a symbol “A” is a triangle obtained from six triangles encircling the point \( p_1 \). In order to explain the way how to obtain Figure 4 in detail, we explain the action of \( G \). In Figure 5, \( G \) fixes \( A \) and \( A' \), and sends other points as \( B \rightarrow C' \rightarrow D, B' \rightarrow C \rightarrow D' \). Let \( L_A \) be the union of two thick triangles which are the union of triangles surrounding \( A \) and \( A' \) and the three arcs connecting these thick triangles. We obtain \( T_B, T_C \) and \( T_D \) in the same way as above. If we remove \( T_A, \ldots, T_D \) from \( \Sigma_3 \), we have six annuli \( I, \ldots, IV \) remained. Each of these annuli are divided into eight thick triangles. In the left of Figure 6, we put \( T_A, \ldots, T_D \) on \( \Sigma_3 \) such that \( G \) acts as a \( 2\pi/3 \) rotation. The annulus \( I \) is as in the middle of Figure 6. We put this triangulated annulus with taking care of the edge correspondence, then we have the right of Figure 6. After we put the other annuli \( II, \ldots, VI \) in the same way, we have Figure 4.

In Figure 4, \( G \) is a \( 2\pi/3 \) rotation about the center of \( A \), and is a periodic map with order 3 whose Dehn twist presentation was obtained by Okuda-Takamura [14].
the left of Figure 7, $F$ exchanges edges $1, 2, \cdots, 7$ clockwise. If we draw these edges $1, 2, \cdots, 7$ on Figure 4, we obtain the right of Figure 7. By this figure, we can observe that the total valency of $F$ is \( \left( \frac{7}{7} \cdot \frac{1}{7} + \frac{2}{7} + \frac{4}{7} \right) \), and $F$ corresponds to $f_{3,9}$ in Theorem 2. The product of Dehn twists $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 5 \cdot 4 \cdot 8$ representing $f_{3,9}$ brings the edges on a figure of $\Sigma_3$ on the upper left of Figure 8 according to their numbering. The upper left of Figure 8 is excerpted from Figure 21 of [5]. Let $\Phi$ be an orientation preserving homeomorphism from the upper left $\Sigma_3$ to the lower left $\Sigma_3$ in Figure 8 which brings the arcs with the same number. Since the complement of these arcs is an open disk in
each $\Sigma_3$, $\Phi$ is uniquely determined up to isotopy. The circles $r_i = \Phi(c_i)$ are as in the lower right of Figure 8, and we see $F = t_{r_6} t_{r_5} t_{r_4} t_{r_3} t_{r_2} t_{r_1} t_{r_5} t_{r_4} t_{r_8}$.

**Proposition 5.** The automorphism group $\text{PSL}_2(7)$ of Klein quartic $\{(x : y : z) \in \mathbb{CP}^2 | x^3 y + y^3 z + z^3 y = 0\}$ is generated by

$$F = t_{r_6} t_{r_5} t_{r_4} t_{r_3} t_{r_2} t_{r_1} t_{r_5} t_{r_4} t_{r_8}, \quad G = t_{q_1} t_{q_2} t_{q_3} t_{q_1} t_{q_2} t_{q_1} t_{q_2},$$

and its defining relations are $F^7 = G^3 = (GF)^2 = (GFG^{-1}F^{-1})^4 = 1$. In the above presentation $r_i$ are simple closed curves shown in Figure 8, and $q_i, q_i'$ are simple closed curves shown in Figure 9.
3.2. A Dehn twist presentation of the action of $S_3 \rtimes (\mathbb{Z}_4 \times \mathbb{Z}_4)$. The finite group action on $\Sigma_3$ with the second largest order is the automorphism group of the Fermat quartic $\{(x : y : z) \in \mathbb{C}P^2 \mid x^4 + y^4 + z^4 = 0\}$. This group is isomorphic to $S_3 \rtimes (\mathbb{Z}_4 \times \mathbb{Z}_4)$, where $S_3 = \langle x, y \mid x^2 = y^3 = 1, xyx^{-1} = y^{-1} \rangle$ acts on $\mathbb{Z}_4 \times \mathbb{Z}_4 = \langle z, w \mid z^4 = w^4 = 1, zw = wz \rangle$ by $xz^{-1} = w, xwx^{-1} = z, yzy^{-1} = w, ywy^{-1} = (zw)^{-1}$.

Figure 10 is obtained by editing Figure 10 of [7]. A fundamental region of the automorphism group of the Fermat quartic is a union of adjacent two triangles. Each thick triangle is a union of three fundamental regions. Figure 11 is obtained from Figure 10 by identifying each pair of edges with the same number. The triangles in Figure 11 correspond to the thick triangles in Figure 10. We obtained Figure 11 in the same way as in the previous subsection. Let $P$ be a clockwise $2\pi/3$ rotation about the center of $c_0$ and $Q$ a clockwise $2\pi/8$ rotation about the center of Figure 10. Then the
automorphism group of the Fermat quartic is generated by $P$ and $Q$. On the left of

Figure 12, the rotation $Q$ maps $e_1 \rightarrow e_2 \rightarrow \cdots \rightarrow e_8 \rightarrow e_1$. If we draw $e_1, \ldots, e_8$ on Figure 11, we obtain the right of Figure 12. The total valency of $Q$ is \( \left( 8, \frac{1}{8} + \frac{1}{4} + \frac{5}{8} \right) \), which corresponds to $f_{3,7}$ of Theorem 2. On upper left of Figure 13 excerpted from Figure 20 of [5], a product of Dehn twists $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 5 \cdot 4 \cdot 3 \cdot 8$ representing $f_{3,7}$ brings $e_i$ to $e_{i+1}$ ($i = 1, \ldots, 6$) and $e_7$ to $e_1$. Let $\Phi$ be an orientation preserving homeomorphism from the upper left of Figure 13 to the lower left of Figure 13 which sends $e_i$ to $e_i$. The circles $s_i = \Phi(e_i)$ are as on the lower right of Figure 13 and we have $Q = t_{s_6} t_{s_5} t_{s_4} t_{s_3} t_{s_2} t_{s_5} t_{s_4} t_{s_3} t_{s_6}$. 
Proposition 6. The automorphism group of the Fermat quartic \( \{(x : y : z) \in \mathbb{C}P^2 \mid x^4 + y^4 + z^4 = 0\} \) is generated by

\[ P = t_{q_1}t_{q_2}t_{q_3}t_{q'_3}, \quad Q = t_{s_6}t_{s_5}t_{s_4}t_{s_3}t_{s_2}t_{s_1}t_{s_8}t_{s_7} \]

and its defining relations are \( P^3 = Q^8 = (PQ)^2 = (PQ^3)^3 = 1 \). In the above presentation, \( s_i \) are simple closed curves shown in Figure 13, and \( q_i \) and \( q'_i \) are those in Figure 14.

3.3. A Dehn twist presentation of the action of \( \mathbb{Z}_2 \ltimes SL_2(3) \). This semi-direct product \( \mathbb{Z}_2 \ltimes SL_2(3) \) is defined by the action of \( \mathbb{Z}_2 = \langle \sigma \mid \sigma^2 = 1 \rangle \) on \( SL_2(3) = \langle \sigma_1, \sigma_2 \mid \sigma_1^3 = 1, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle \) by \( \sigma_1\sigma = \sigma_2 \). If we put \( T = \sigma_1, S = \sigma, \) then \( \mathbb{Z}_2 \ltimes SL_2(3) = \langle T, S \mid T^3 = S^2 = 1, (TS)^3 = (ST)^3 \rangle \). In this group \( TS \) has order 12 and corresponds to \( f_{3,6} \) of Theorem 2. Figure 15 obtained by modifying Figure 24 of [5] shows the action on \( \Sigma_3 \) of the product of Dehn twists \( t_{c_6}t_{c_5}t_{c_4}t_{c_3}t_{c_2}t_{c_8} \) representing
Let $e_i$ be an edge with single arrow and index $i$, $E_i$ be an edge with double arrow and index $i$, and $\bar{e}_i$, $\bar{E}_i$ be the edges with opposite orientation of $e_i$, $E_i$ respectively. The group $\mathbb{Z}_2 \ltimes SL_2(3)$ acts on a graph in $\Sigma_3$ consists of these edges. We regard the left end $F$ of the edge $E_0$ as the fundamental domain of this action and, for an elements $g$ of $\mathbb{Z}_2 \ltimes SL_2(3)$, the end of an edge marked by the symbol $g$ is $Fg$. The action of the involution $S$ on these edges is as follows, $E_{11} \rightarrow \bar{e}_1$, $E_0 \rightarrow \bar{E}_0$, $E_1 \rightarrow e_0$, $E_2 \rightarrow e_4$, $E_3 \rightarrow \bar{E}_3$, $E_4 \rightarrow e_3$, $E_5 \rightarrow \bar{e}_7$, $E_6 \rightarrow \bar{E}_6$, $E_7 \rightarrow e_6$, $E_8 \rightarrow e_{10}$, $E_9 \rightarrow \bar{E}_9$, $E_{10} \rightarrow e_9$, $e_2 \rightarrow \bar{e}_8$, $e_5 \rightarrow e_{11}$. For the circles $\gamma_1, \ldots, \gamma_6$, $\delta$ and $\epsilon$ on the left of Figure 16,
by investigating their intersection with edges $e_i$ and $E_i$ and the action of $S$ on these edges, we see that $S$ sends $\gamma_1, \gamma_2, \gamma_3$ to $\gamma_6, \gamma_5, \gamma_4$ respectively, and fixes $\delta$ and $\epsilon$ setwisely with reversing their orientations. There is an orientation preserving homeomorphism $\Phi$ between $\Sigma_3$ on the upper right of Figure 16 to $\Sigma_3$ on the left of Figure 16 sending the circles $\gamma_1, \ldots, \gamma_6, \delta$ and $\epsilon$ to the circles with the same names. Korkmaz [10] showed that $\sigma = t_2^2 t_2 t_2^2 t_2 t_2^2 t_2$, where Dehn twists are about the circles on the lower right of Figure 16. These circles $d_1, d_2, d_3$ are obtained from $\epsilon$ by the maps $t_1^2 t_2^2 t_2^2 t_2 t_2^2 t_2$ respectively. We denote the images of the circles $a, b, d_0 = \epsilon, d_1, d_2, d_3$ by the map $\Phi$ by the same symbols and show them in Figure 17.

In summary, we have:

**Proposition 7.** The action of $\mathbb{Z}_2 \ltimes SL_2(3) = \langle T, S \mid T^3 = S^2 = 1, (TS)^3 = (ST)^3 \rangle$ on $\Sigma_3$ is generated by $TS = t_0 t_1 t_2 t_3 t_4 t_5 t_6 t_7$, $S = t_2 t_3 t_4 t_5 t_6 t_7 t_8 t_9$. In the above presentation, $a, b, c, d_0, d_1, d_2, d_3$ are simple closed curves shown in Figure 17.

![Figure 17](image-url)
obtained independently from the Hasegawa’s presentations. The hyperelliptic mapping class group of \( \Sigma_3 \) is generated by Dehn twists about the circles \( c_1, c_2, \ldots, c_7 \). In our presentations, we use Dehn twists about these circles. In the following presentation, \( k \) means the Dehn twist \( t_{c_k} \) about the simple closed curve \( c_k \) and \( \overline{k} \) means \( t_{c_k}^{-1} \).

3.4.1. A Dehn twist presentation of the action of \( D_{2,12,5} \). The group \( D_{2,12,5} \) preserves a graph on \( \Sigma_3 \) illustrated in Figure 18. The edge with 1 is the fundamental domain \( F \) of this action and the edge with \( g \in D_{2,12,5} \) is \( Fg \). We denote the curve with edge \( x y^i \) by \( a_i \) (\( i = 0, 1, \ldots, 5 \)) and the curve with opposite orientation by \( \overline{a}_i \). The element \( x \in D_{2,12,5} \) maps \( a_0 \to \overline{a}_0, a_1 \to a_5, a_2 \to a_4, a_3 \to a_3, \) and the element \( y \in D_{2,12,5} \) maps \( a_0 \to a_1 \to a_2 \to a_3 \to a_4 \to a_5 \to a_0 \to a_1 \to \cdots \). By investigating the actions of Dehn twists on these circles, we see:

![Figure 18](image-url)

**Figure 18**

**Proposition 8.** The action of \( D_{2,12,5} \) on \( \Sigma_3 \) is generated by \( x = (1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6) \cdot (1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6) \cdot (1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6) \), and \( y = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \).

3.4.2. A Dehn twist presentation of the action of \( \mathbb{Z}_2 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_8) \). The group \( \mathbb{Z}_2 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_8) \) preserves a graph on \( \Sigma_3 \) illustrated in Figure 19. The edge with 1 is the fundamental domain \( F \) of this action and the edge with \( g \in \mathbb{Z}_2 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_8) \) is \( Fg \). We denote the curve with edge \( z^i \) by \( b_i \) (\( i = 0, 1, \ldots, 7 \)) and the curve with opposite orientation by \( \overline{b}_i \). The element \( x \in \mathbb{Z}_2 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_8) \) maps \( b_0 \to \overline{b}_0, b_1 \to b_7, b_2 \to b_6, b_3 \to b_5, b_4 \to b_4, \) and the element \( z \in \mathbb{Z}_2 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_8) \) maps \( b_0 \to b_1 \to b_2 \to b_3 \to b_4 \to b_5 \to b_6 \to b_7 \to b_0 \). By investigating the actions of Dehn twists on these circles, we see:

**Proposition 9.** The action of \( \mathbb{Z}_2 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_8) \) on \( \Sigma_3 \) is generated by \( x = (1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7) \cdot (1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7) \cdot (1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7) \), and \( z = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \).

3.4.3. A Dehn twist presentation of the action of \( \mathbb{Z}_2 \times S_4 \). The action of the group \( S_4 \) preserves a graph on \( \Sigma_3 \) illustrated in Figure 20. The edge with 1234 is the fundamental
domain $F$ of this action and the edge with $a_1a_2a_3a_4$ is $F\sigma$ for $\sigma \in S_4$ such that $\sigma(i) = a_i$ for each $i \in \{1, 2, 3, 4\}$. We denote by $\tilde{d}_i$ the circle having an arrow with the symbol $d_i$ and by $\bar{d}_i$ the circle with opposite orientation. The element $x \in \mathbb{Z}_2 \times S_4$ acts on $\Sigma_3$ as a hyperelliptic involution. The cyclic permutation $y = (2, 3, 4) \in \mathbb{Z}_2 \times S_4$ maps $d_1 \to d_5 \to d_4 \to d_1$, $d_2 \to d_3 \to \bar{d}_6 \to d_2$, and the cyclic permutation $z = (1, 4, 3, 2) \in \mathbb{Z}_2 \times S_4$ maps $d_1 \to d_2 \to d_1$, $d_3 \to d_4 \to \bar{d}_5$, $d_5 \to d_6 \to d_5$. By investigating the actions of Dehn twists on these circles, we see:

**Figure 19**

**Proposition 10.** The action of $\mathbb{Z}_2 \times S_4$ on $\Sigma_3$ is generated by $x = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, $y = 5 \cdot 6 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 2 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ and $z = (1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7)^2$.

4. A list of non maximal finite group actions on $\Sigma_3$

In this section, we list non maximal finite group actions on $\Sigma_3$ as subgroup actions of maximal finite group actions. This list is obtained by using GAP 4. In this list, 3.xx is a name of a finite group action on the list by Broughton [2], especially, 3.at = $PSL_2(7)$ (§2.1), 3.as = $S_3 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4)$ (§2.2), 3.ao = $Z_2 \ltimes SL_2(3)$ (§2.3), 3.ap = $Z_2 \times S_4$, 3.am.1 = $Z_2 \ltimes (Z_2 \times Z_6)$, 3.ah = $D_{2, 12, 5}$ (§2.4), 3.aa = $Z_{14}$, 3.t = $Z_9$ (§1).

3.xx : 3.yy $\varsubsetneq F_1 = \ast \ast \ast$, $F_2 = \ast \ast \ast$, [\ldots]
means that $3.xx$ is a subgroup action of $3.yy$, the elements $F_1, F_2$ of $3.yy$ generate the action of $3.xx$, and, in $\ldots$, are defining relations among $F_1, F_2$.

3.aq : $3.a\supset F_1 = P, F_2 = Q^{-2}, [F_3^2, F_2, F_2^{-1}F_1^{-1}F_2^{-1}F_1, F_2^{-1}F_1F_2F_1^{-1}F_2^{-1}F_1]$
3.am.2 : $3.a\supset F_1 = Q, F_2 = PQ^{-1}P, [F_2, F_2F_2^{-1}F_2F_1, F_2F_2F_2F_2F_2F_2F_2F_1]$
3.al : $3.at \supset F_1 = F^{-3}GF^{-2}, F_2 = FG^3F, [F_1^3, F_2F_1F_2F_1, F_2^2, F_2F_2F_2F_2F_1, F_2F_2F_2F_2F_2F_1]$
3.as \supset F_1 = QP, F_2 = PQ^{-1}P,
$[F_1^2, F_2^2, F_3^2, F_1F_2F_3F_2^{-1}, F_3F_1F_2F_2^{-1}F_1, F_3F_1F_2F_2F_3F_1]$
3.ap \supset F_1 = yx^{-1}, F_2 = z, $[F_2, F_2^2, F_2F_1F_2F_1F_2F_2F_2F_1]$
3.ak : $3.ap \supset F_1 = y, F_2 = z, [F_2^2, F_2^2, F_2F_1F_2F_1F_2F_2F_2F_1]$
3.aj : $3.ao \supset F_1 = T, F_2 = STS^{-1}, [F_3^2, F_2F_2F_2F_1^{-1}F_2^{-1}F_1^{-1}]$
3.ai : $3.ap \supset F_1 = x, F_2 = z, F_3 = zzy^{-1}, [F_3^2, F_2^2, F_3^2, F_2F_1F_2F_1, F_3F_1F_3^{-1}F_1, F_3F_2F_3F_2F_3]$
3.ag : $3.at \supset F_1 = F, F_2 = GF^{-3}G^{-1}FG^{-1}, [F_2^3, F_2F_2F_2^{-1}F_2^{-1}, F_2F_2F_2F_2F_2F_1, F_2]$  
3.ad.1 : $3.ap \supset F_1 = x, F_2 = y, F_3 = zyz, [F_2^2, F_2^2, F_2F_2F_2F_1, F_3F_1F_3^{-1}F_1, F_3F_2F_3F_2F_3]$
3.am.1 \supset F_1 = x, F_2 = y, F_3 = z^{-2}, $[F_2^2, F_2^2, F_1F_3^{-1}F_1, F_2F_2F_2F_2F_1, F_3F_2F_2F_2F_2F_3]$
3.ad.2 : $3.ao \supset F_1 = S, F_2 = TST^{-1}, F_3 = T^{-1}ST, [F_2^3, F_2^3, F_1F_3F_2F_2F_2, F_2F_1F_2F_3F_2F_3]$  
3.ac.1 : $3.as \supset F_1 = Q^{-2}, F_2 = PQ^{-2}P^{-1}, [F_2F_1^{-1}F_3F_1, F_2^3, F_2]$  
3.ac.2 : $3.am.1 \supset F_1 = xz^{-1}, F_2 = z^{-1}x^{-1}, [F_2^3, F_2^3, F_1F_2^{-1}F_2^{-1}F_1F_2^{-1}F_1^{-1}F_2^{-1}F_2^{-1}F_1]$
3.ab.1 : $3.am.1 \supset F_1 = y, F_2 = z, [F_2^2, F_2F_1F_2^{-1}F_1, F_2^2]$
3.ab.2 : $3.as \supset F_1 = Q^{-1}P, F_2 = Q^3P, [F_2F_2F_2^{-1}F_2F_1, F_2^3, F_2^3]$
3.z : $3.at \supset F_1 = G^{-1}F^{-1}, F_2 = F^{-3}GF^{-2}, [F_2^2, F_2F_1^{-1}F_2^{-1}F_1, F_2F_2F_2F_1, F_2]$  
3.as \supset F_1 = PQ^2, F_2 = PQ^{-1}P^{-1}Q^{-2}P^{-1}, [F_1^3, F_2, F_2F_1F_2F_1F_2, F_2F_1F_2F_2F_1, F_2F_2F_2F_2F_1]$
3.ap \supset F_1 = z, F_2 = yzy^{-1}, [F_1^3, F_2^3, F_2F_1F_2F_1]$
3.y : $3.ap \supset F_1 = x, F_2 = z, F_3 = yzyz^{-1}y^{-1}, [F_2^3, F_2^3, F_2F_2F_2F_2F_2F_2F_2F_2F_1, F_3F_3F_3F_2F_2F_2F_1, F_3F_3F_3F_2F_2F_2F_1]$
3.ah \supset F_1 = y^{-2}, F_2 = y^2x^{-1}y^{-6}, [F_2^2, F_2F_1F_2F_1F_1, F_2^6]$
3.x : $3.ah \supset F_1 = yx^{-1}, F_2 = y^{-1}x^{-1}, [F_2F_2F_1, F_2F_2F_1, F_2F_2F_1, F_2F_2F_1F_2F_2F_1]$
3.v : $3.ao \supset F_1 = T, F_2 = STS^{-1}TS^{-1}, [F_3^3, F_2F_1F_2^{-1}F_1, F_1F_1^3]$  
3.u : $3.ah \supset F_1 = y, [F_1^2]$  
3.s.1 : $3.as \supset F_1 = Q^{-1}PQ^{-2}, F_2 = Q^{-3}P, [F_2F_2^{-1}F_2^{-1}F_1^{-1}, F_2^3, F_2^3F_1^3, F_2^2F_1^{-2}]$
3.so \supset F_1 = STS^{-1}T^{-1}, F_2 = ST^{-1}S^{-1}T, [F_2F_1F_2F_1^{-1}F_1^{-1}F_2F_1F_2F_1]$
3.s.2 : $3.am.1 \supset F_1 = z^{-2}y^{-1}, F_2 = z^{-2}y^{-1}x^{-2}, [F_2^2, F_2F_1F_2F_2F_1]$  
3.ap \supset F_1 = y, F_2 = zyzx^{-1}, [F_2^2, F_2F_1F_2F_2F_1, F_2^2]$  
3.r.1 : $3.ap \supset F_1 = x, F_2 = y, F_3 = zyz^{-1}y^{-1}z,$
$[F_1^3, F_2^3, F_2F_2F_2F_3F_2F_1, F_2F_2F_2F_2F_3F_1, F_2F_2F_2F_2F_3F_1, F_2F_2F_2F_2F_3F_1]$  
3.am.1 \supset F_1 = x, F_2 = y, F_3 = y^{-1}zxxz,
$[F_2^2, F_2^2, F_2F_3F_1F_3F_3F_2F_3F_2F_2F_2F_2F_2F_1F_2F_2F_2F_2F_2F_2F_3F_1]$  

\textsuperscript{(1)}its branching indices is (3,3,7)
3.2: 3.at ⊃ F₁ = F⁻¹GF⁻¹GF, F₂ = GF⁻²GF⁻², [F₁, F₂, F₁F₂F₁F₂F₁F₂F₁F₂F₁F₂]
3.as ⊃ F₁ = PQ⁻¹P, F₂ = PQ⁻³P, [F₁, F₂, F₁F₂⁻¹F₁F₂⁻¹]
3.ao ⊃ F₁ = TSTS⁻¹T, F₂ = T⁻¹ST,, [F₁, F₂, F₁F₂⁻¹F₁F₂F₁F₂F₁F₂]
3.am.1 ⊃ F₁ = x, F₂ = yz², [F₁, F₂, F₁F₂⁻¹F₁F₂, F₁]
3.q.1 (x, x, y⁻¹, y): 3.ap ⊃ F₁ = x, F₂ = z⁻¹y⁻¹, [F₁, F₂F₁F₂⁻¹F₁F₂]
3.am.1 ⊃ F₁ = y, F₂ = z⁻², [F₁, F₂⁻¹F₁F₂F₁, F₂]
3.q.1 (x, xy⁻¹, y): 3.as ⊃ F₁ = Q⁻²P, F₂ = PQ⁻¹P, [F₁, F₂, F₁F₂F₁F₂⁻¹]
3.ao ⊃ F₁ = S, F₂ = TSTS⁻¹T, [F₁, F₂, F₁F₂F₁F₂⁻¹]
3.q.1 (x, y², x, y): 3.ah ⊃ F₁ = yxy, F₂ = y³, [F₁, F₂, F₁F₂⁻¹F₁F₂]
3.am.1 ⊃ F₁ = x, F₂ = z⁻², [F₁, F₂, F₁F₂F₁F₂F₁F₂]
3.p (x⁶, x, x): 3.am.1 ⊃ F₁ = z, [F₁]
3.o (x, x, x²): 3.an ⊃ F₁ = x⁻³, [F₁]
3.q.2 (x, y², x, y): 3.at ⊃ F₁ = F, [F₁]
3.m.3: 3.at ⊃ F₁ = G⁻¹F⁻¹, F₂ = G⁻¹FG⁻¹FG⁻¹G⁻¹F, [F₁, F₂, F₁F₂F₁F₂]
3.as ⊃ F₁ = QP, F₂ = Q⁻¹PQ⁻³P⁻¹, [F₁, F₂, F₁F₂F₁F₂]
3.am.1 ⊃ F₁ = z, F₂ = yzy⁻¹y⁻¹x⁻¹, [F₂, F₁F₂F₁F₂F₁F₂⁻¹]
3.ah ⊃ F₁ = x, F₂ = yx⁻¹y⁻¹x⁻¹, [F₂, F₁F₂⁻¹F₁F₂]
3.k: 3.ao ⊃ F₁ = TSTS⁻¹ST⁻¹, [F₁]
3.j: 3.ap ⊃ F₁ = x, F₂ = z, [F₁, F₂, F₂F₁F₂⁻¹F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F₁F
an action which is not in the list by Broughton an action of \( \langle b, c \mid b^2 = c^4 = 1, bcb = c^{-1} \rangle \) (order = 16)

\[
3. \text{as} \ni F_1 = Q^{-2}, F_2 = PQ^{-1}P, F_3 = PQ^3P,
\]
\[
[F_2^2, F_3^{-1}F_1F_3^{-1}F_1, F_3^4, F_3^{-2}F_2^2, F_2F_1^{-1}F_1F_2, F_3F_2F_3F_2]
\]

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