Periodic maps on surfaces and examples of Lefschetz fibrations

廣瀬 進 (Susumu Hirose) *

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*佐賀大学 (Saga University)
Definition of Lefschetz fibration

$M$ : compact oriented $C^\infty$ 4-manifold
A $C^\infty$ map $f : M \to S^2$ is a genus $g$ Lefschetz fibration

$$\iff_{\text{def}}$$

(1) $df : TM \to TS^2$ is surjective except at several points $p_1, \ldots, p_k$,
(2) in a neighborhood of $p_i$, $f(z_1, z_2) = z_1^2 + z_2^2$,
(3) there is no $-1$ sphere in each fiber,
(4) its general fiber $= \Sigma_g$.

$q_i := f(p_i)$.
Assume $q_i \neq q_j$. 

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Find examples of non-holomorphic Lefschetz fibrations.

Examples: Fuller, Endo, Korkmaz-Ozbagci, etc...
\( t_i := \) the right handed Dehn twist about \( \tau_i \)

\( W = t_1 t_2 \cdots t_r = id \) (in the mapping class group of \( \Sigma_g \)) \( \cdots \)

Positive relation

Positive relation \( W = id \Leftrightarrow \text{L.f. } f : M \to S^2 =: Lf(W) \)
A way to find positive relations

Use periodic maps on $\Sigma_g$

An orientation preserving diffeomorphism $f : \Sigma_g \to \Sigma_g$

is a periodic map

def $\iff$ There is a positive integer $n$ so that $f^n = id_{\Sigma_g}$.

The minimum of $n$ is the period of $f$.

$f$ is isotopic to $t_1 t_2 \cdots t_k$

(positive Dehn twist presentation for $f$) $\Rightarrow$

$t_1 t_2 \cdots t_k \cdot t_1 t_2 \cdots t_k \cdots t_1 t_2 \cdots t_k$ is isotopic to $f^n = id$. $n$ times

Find positive Dehn twist presentations for periodic maps.
**Valency data**

\[ f : \Sigma_g \rightarrow \Sigma_g \] is a periodic map of period \( n \) \( \Rightarrow \)

\[ p_f : \Sigma_g \rightarrow \Sigma_g/f \] is an \( n \)-fold branched covering.

\( B_f := \) the set of branch point of \( p_f \).

Describe the rotation around the preimages of the point in \( B_f \):

Let \( B_f = \{ b_1, b_2, \ldots, b_k \} \). Take a small disk around \( b_i \).

The valency of \( b_i := \theta_i/n \). The valency data (notation by Ashikaga and Ishizaka) of \( f := (n, \theta_1/n + \cdots + \theta_k/n) \), where the notation + is just a symbol do not add. (If you add these rational numbers, you get an integer.)
Nielsen’s description

$f_1, f_2 : \Sigma_g \to \Sigma_g$ are diffeomorphisms
$f_1$ and $f_2$ are conjugate $\iff \exists$ a self diffeomorphism $g$ over $\Sigma_g$
\[
\begin{array}{c}
\Sigma_g \xrightarrow{f_1} \Sigma_g \\
g \\
\Sigma_g \xrightarrow{f_2} \Sigma_g
\end{array}
\]
such that
\[
\begin{array}{c}
g \\
\Sigma_g \\
g
\end{array}
\]

**Theorem [Nielsen]**
The conjugacy class of periodic map on $\Sigma_g$ is determined by the valency data $(n, \theta_1/n + \cdots + \theta_k/n)$. 
Example for $\Sigma_2$

Introduction
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Genus 3
Genus 4
Conclusion remarks
Example for $\Sigma_2$

A periodic map on $\Sigma_2$
A periodic map on $\Sigma_2 \cdots (10, \frac{3}{10} + \frac{1}{5} + \frac{1}{2})$
A periodic map on $\Sigma_2$ ⋯ ($10, 3/10 + 1/5 + 1/2$)
Example for $\Sigma_2$

A periodic map on $\Sigma_2 \cdots (10, 3/10 + 1/5 + 1/2)$
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A periodic map on $\Sigma_2$
Example for $\Sigma_2$

A periodic map on $\Sigma_2$  \( \cdots (10, \frac{3}{10} + \frac{1}{5} + \frac{1}{2}) \)

A periodic map on $\Sigma_2$  \( \cdots (8, \frac{1}{8} + \frac{3}{8} + \frac{1}{2}) \)
Example for $\Sigma_2$

A periodic map on $\Sigma_2$ · · · (10, $3/10 + 1/5 + 1/2$)

A periodic map on $\Sigma_2$ · · · (8, $1/8 + 3/8 + 1/2$)

We can generalize this construction to the higher genus. These maps have the maximal and secondary maximal periods for each genus.
Any periodic map on $\Sigma_3$ is (conjugate to) the power of the following maps: (hyperelliptic, non-hyperelliptic)

(14, $1/14 + 3/7 + 1/2$)
(12, $1/12 + 5/12 + 1/2$)
(12, $1/12 + 1/4 + 2/3$)
(9, $1/9 + 1/3 + 5/9$)
(8, $1/8 + 1/8 + 3/4$)
(8, $1/8 + 1/4 + 5/8$)
(7, $1/7 + 2/7 + 4/7$)
(4, $1/2 + 1/2$)
(2, ).
Any periodic map on $\Sigma_3$ is (conjugate to) the power of the following maps: (hyperelliptic, non-hyperelliptic)

$$(14, \frac{1}{14} + \frac{3}{7} + \frac{1}{2}) = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1,$$

$$(12, \frac{1}{12} + \frac{5}{12} + \frac{1}{2}) = 6 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1,$$

$$(12, \frac{1}{12} + \frac{1}{4} + \frac{2}{3}) = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 8,$$

$$(9, \frac{1}{9} + \frac{1}{3} + \frac{5}{9}) = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 8,$$

$$(8, \frac{1}{8} + \frac{1}{8} + \frac{3}{4}) = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1,$$

$$(8, \frac{1}{8} + \frac{1}{4} + \frac{5}{8}) = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 5 \cdot 4 \cdot 3 \cdot 8,$$

$$(7, \frac{1}{7} + \frac{2}{7} + \frac{4}{7}) = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 5 \cdot 4 \cdot 8,$$

$$(4, \frac{1}{2} + \frac{1}{2}) = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot (7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)^3,$$

$$(2,) = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot (7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)^5.$$

Dehn twist presentations for these maps are as above.
Genus 3 Lefschetz fibration

$W$: a word of right handed Dehn twists,

$L_{f_{D^2}}(W)$: a Lefschetz fibration over $D^2$ determined by $W$.

$(8, 1/8 + 1/8 + 3/4) = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 =: W_1$

$\Rightarrow W_1^8 = id$ is a positive relation.

$(12, 1/12 + 1/4 + 2/3) = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 8 =: W_2$

$\Rightarrow W_2^{12} = id$ is a positive relation.

$(8, 1/8 + 1/8 + 3/4)^4 = (12, 1/12 + 1/4 + 2/3)^6 \Rightarrow$

$\exists F$: a fiber preserving diffeo. $\partial L_{f_{D^2}}(W_1^4) \rightarrow -\partial L_{f_{D^2}}(W_2^6)$.

$L_{f}(W_1^4 \# W_2^6) := L_{f_{D^2}}(W_1^4) \cup_F L_{f_{D^2}}(W_2^6)$. 
We will show that $L_f(W_1^4 \# W_2^6)$ is non-holomorphic by the method of Endo and Nagami [Trans. AMS, 357(2004)].

\[
\begin{align*}
T & \quad \text{Theorem} \quad [\text{Endo, Math. Ann. 316 (2000)}] \\
& \quad L_f: \text{a genus } g \text{ Lefschetz fibration,} \\
& \quad (1) \ L_f \text{ is hyperelliptic, (2) all sing. fiber of } L_f \text{ is irreducible} \\
\Rightarrow & \quad \text{the signature of } L_f \\
& \quad \text{the number of singular fibers of } L_f = -\frac{g + 1}{2g + 1}.
\end{align*}
\]

$n := \text{the number of singular fibers of } L_f(W_1^4 \# W_2^6)$

$= \text{the word length of } W_1^4 + \text{the word length of } W_2^6 = 64$

$\sigma := \text{the signature of } L_f(W_1^4 \# W_2^6)$

$= \text{the signature of } L_f(W_1^4) + \text{the signature of } L_f(W_2^6) \text{ (by Novikov additivity)}$

$= (-16) + (-20) \text{ (by calculation using Meyer cocycle)} = -36$

$\Rightarrow \sigma/n = -9/16 \neq -4/7 \Rightarrow L_f(W_1^4 \# W_2^6) \text{ is not hyperelliptic.}$
The slope $\lambda$ of Lefschetz fibration := $12 - \frac{4}{1 + \frac{\sigma}{n}}$.

**Theorem** [Konno, O.J.M. 28(1991)]

$Lf$: a genus 3 Lefschetz fibration,
(1) $Lf$ is non-hyperelliptic, (2) isotopic to holomorphic fibration
$\Rightarrow \lambda \geq 3$.

For our $Lf(W_1^4 \# W_2^6)$, $\lambda = 12 - \frac{4}{1 + \frac{-36}{64}} = \frac{20}{7} \leq 3$.
$\Rightarrow Lf(W_1^4 \# W_2^6)$ is non-holomorphic.
Any periodic map on $\Sigma_4$ is (conjugate to) the power of the following maps: (hyperelliptic, non-hyperelliptic)

$(18, 1/18 + 4/9 + 1/2)$, $(16, 1/16 + 7/16 + 1/2)$,
$(15, 1/15 + 1/3 + 3/5)$, $(12, 1/12 + 1/6 + 3/4)$,
$(12, 1/12 + 1/3 + 7/12)$
$(10, 1/10 + 1/10 + 4/5)$
$(10, 2/5 + 1/2 + 1/2 + 3/5)$, $(10, 1/10 + 3/10 + 3/5)$,
$(6, 1/6 + 1/3 + 2/3 + 5/6)$, $(6, 1/3 + 1/3 + 1/3 + 1/2 + 1/2)$,
$(6, 1/2 + 1/2)$, $(5, 1/5 + 2/5 + 3/5 + 4/5)$
Any periodic map on $\Sigma_4$ is (conjugate to) the power of the following maps: (hyperelliptic, non-hyperelliptic)

$(18, 1/18 + 4/9 + 1/2)$, $(16, 1/16 + 7/16 + 1/2)$,
$(15, 1/15 + 1/3 + 3/5)$, $(12, 1/12 + 1/6 + 3/4)$,
$(12, 1/12 + 1/3 + 7/12) = 6 \cdot 5 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 11$

$(10, 1/10 + 1/10 + 4/5) = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$(10, 2/5 + 1/2 + 1/2 + 3/5)$, $(10, 1/10 + 3/10 + 3/5)$,
$(6, 1/6 + 1/3 + 2/3 + 5/6)$, $(6, 1/3 + 1/3 + 1/3 + 1/2 + 1/2)$,
$(6, 1/2 + 1/2)$, $(5, 1/5 + 2/5 + 3/5 + 4/5)$

Dehn twist presentation for two of these maps are as above. For the list of presentations for all maps, please see my preprint available from http://www.ms.saga-u.ac.jp/~hirose/work.html
Genus 4 Lefschetz fibration

\[(12, 1/12 + 1/3 + 7/12) = 6 \cdot 5 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 11 =: W_3,\]
\[(10, 1/10 + 1/10 + 4/5) = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 =: W_4,\]
\[(12, 1/12 + 1/3 + 7/12)^6 = (10, 1/10 + 1/10 + 4/5)^5 \Rightarrow \exists \text{ a fiber preserving diffeo. } F : \partial Lf_D^2(W_3^6) \to -\partial Lf_D^2(W_4^5).\]

\[Lf(W_3^6 \# W_4^5) := Lf_D^2(W_3^6) \cup_F Lf_D^2(W_4^5).\]

\[n(Lf(W_3^6 \# W_4^5)) = 105,\]
\[\sigma(Lf(W_3^6 \# W_4^5)) = Lf(W_3^6) + Lf(W_4^5) \text{ (by Novikov)} \]
\[= (-32) + (-25) \text{ (by Meyer cocycle)} = -57.\]

\[\frac{\sigma}{n} = -\frac{19}{35} \neq -\frac{4+1}{2\cdot 4+1} \Rightarrow Lf(W_3^6 \# W_4^5) \text{ is non-hyperelliptic (by Endo).}\]
**Theorem** [Z. Chen, Inter. J. Math. 4(1993)]

$L_f$: a genus 4 Lefschetz fibration,

1. $L_f$ is non-hyperelliptic,
2. isotopic to holomorphic fibration

$\Rightarrow \lambda := 12 - \frac{4}{1 + \frac{\sigma}{n}} \geq \frac{24}{7}.$

For $L_f(W_3^6 \# W_4^5)$, $\lambda = \frac{13}{4} \leq \frac{24}{7}$. $\Rightarrow L_f(W_3^6 \# W_4^5)$ is non-holomorphic.
Methods to find Dehn twists presentations

**hyperelliptic:** presentations are obtained in [Ishizaka, Rev. Mat. Complut.20 (2007)].

**non-hyperelliptic:** depend on maps.

1. found by using monodromies of plane curve singularity: genus=3, \((12, 1/12 + 1/4 + 2/3), (9, 1/9 + 1/3 + 5/9)\), genus=4, \((15, 1/15 + 1/3 + 3/5)\).
2. found by using computer: genus=3, \((8, 1/8 + 1/4 + 5/8), (7, 1/7 + 2/7 + 4/7)\), genus=4, \((12, 1/12 + 1/3 + 7/12), (10, 1/10 + 3/10 + 3/5)\).
3. found by reducing to the lower genus: genus=4, \((6, 1/6 + 1/3 + 2/3 + 5/6), (6, 1/3 + 1/3 + 1/3 + 1/2 + 1/2), (6, 1/2 + 1/2), (5, 1/5 + 2/5 + 3/5 + 4/5)\).
4. found by hand: genus=4, \((12, 1/12 + 1/6 + 3/4)\).
Coda 1 – signature of periodic map

\[ f : \text{a periodic map on } \Sigma_g \text{ of period } n. \]

The quotient space

\[
\frac{\Sigma_3 \times D^2}{(x, y) \sim (f^{-1}(x), e^{2\pi i/n}y)}
\]

has quotient singularities

\[ \downarrow \text{Hirzebruch-Jung resolution} \]

A 4-manifold (possibly with rational (-1)-curves) with a boundary.

\[ \downarrow \text{blow-downs} \]

A 4-manifold \( M(f) \) with the same boundary.
Example \( f = (12, 1/12 + 1/4 + 2/3) \) of \( \Sigma_3 \)
Example $f = (12, 1/12 + 1/4 + 2/3)$ of $\Sigma_3$
Example \( f = (12, 1/12 + 1/4 + 2/3) \) of \( \Sigma_3 \)
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Example \( f = (12, 1/12 + 1/4 + 2/3) \) of \( \Sigma_3 \)

**Signature** of \( f := \) the signature of \( M(f) \). (Example: \( = 0 \).)

**Problem** Let \( f \) be a periodic map of \( \Sigma_g \) such that \( \Sigma_g/f \) is a 2-sphere with 3 branch points. Is there a right handed Dehn twist presentation \( W \) of \( f \) such that the signature of \( f = \) the signature of \( Lf_{D^2}(W) \)?
Coda 2 – parity of periodic map

$\Sigma_g \hookrightarrow S^4$: standardly embedded.

The orientation preserving diffeomorphism $\phi$ of $\Sigma_g$ is extendable if and only if $\phi$ leaves fixed an even spin-structure on $\Sigma_g$.

**Theorem** [H, A.G.T. 2(2002)]

Any orientation preserving diffeomorphism $\phi$ of $\Sigma_g$ leaves fixed some spin-structure on $\Sigma_g$. 

**Theorem** [Atiyah, Ann. Sci. Éc. Norm. Sup. 4 ser. 4 (1971)]
Parity of $\phi := \begin{cases} 
\text{even} & \text{if } \phi \text{ leaves some even spin-structure fixed and no odd spin-structure fixed} \\
\text{odd} & \text{if } \phi \text{ leaves some odd spin-structure fixed and no even spin structure fixed} \\
\text{neutral} & \text{if } \phi \text{ leaves some odd spin-structure fixed and also some even spin-structure fixed} 
\end{cases}$

Problem

For periodic maps on $\Sigma_g$, find a formula for parity in terms of valency data.