An algorithm to compute harmonic numbers

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Contents

Definition of Harmonic Numbers

- H(n)=harmonic mean of positive divisors of an integer n.
- An integer n is called harmonic when H(n) is integral.

- 1 is called a trivial harmonic number.
- Example. 6 is harmonic.

$$H(6) = \frac{4}{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{6}} = \frac{4 \times 6}{6 + 3 + 2 + 1} = 2$$

Ore’s Conjecture

In 1948, Ore proved that

every perfect number is harmonic.

Ore’s conjecture:

All nontrivial harmonic numbers are even. (??)

if true

There does not exist an odd perfect number!

Known Facts (1)

G. L. Cohen listed all harmonic numbers less than 2×10^9 (130 numbers).

n	H(n)	n	H(n)	n	H(n)	n	H(n)	n	H(n)
1	1	18600	15	360360	44	2290260	41	15495480	86
6	2	18620	14	539400	44	2457000	60	16166592	51
28	3	27846	17	695520	29	2845800	51	17428320	96
140	5	30240	24	726180	46	4358600	37	18154500	75
270	6	32760	24	753480	39	4713984	48	23088800	70
496	5	55860	21	950976	46	4754880	45	23569920	80
672	8	105664	13	1089270	17	5772200	49	23963940	99
1638	9	167400	19	1421280	42	6051500	50	27027000	110
2970	11	173600	27	1539720	47	8506400	49	29410290	81
6200	10	237510	25	2178540	47	8872200	53	32997888	84
8128	7	242060	29	2178540	54	11981970	77	33550336	13
8190	15	332640	26	2229500	35	14303520	86	……………..	……

Known Facts (2)

Problem

Which values does the harmonic mean take? Are there integers n satisfying
$H(n)=4,12,16,18,20,22,\ldots$?

Theorem (Kanold)

For any positive integer c, there exist only finitely many numbers n satisfying $H(n)=c$.

Known Facts (3)

Cohen listed

All harmonic numbers n satisfying H(n) <14.

H(n)	n	H(n)	n
1	1	8	672
2	6	9	1638
3	28	10	6200
5	140	11	2970
	496	13	105664
6	270		33550336
7	8128		

There does not exist a harmonic number n with H(n)=4 or 12

.

Results

We give an algorithm to find all integers n satisfying H(n)=c for a given integer c.

↓ Using a personal computer

We listed all harmonic numbers satisfying H(n)<1000 (there are 1138 such numbers).

↓ In particular

If n is harmonic and 1<H(n)<1000, then n is even.

General Algorithm

Step 1. List the possibilities of the numbers of distinct primes dividing n.

Step 2. List the possibilities of the type of exponents.

We say that n has the type of exponent (a,b,...,z) when the factorization of n is p^a q^b ... r^z.

Step 3. List the possibilities of smallest prime factors of n.

Improved Algorithm

Proposition.

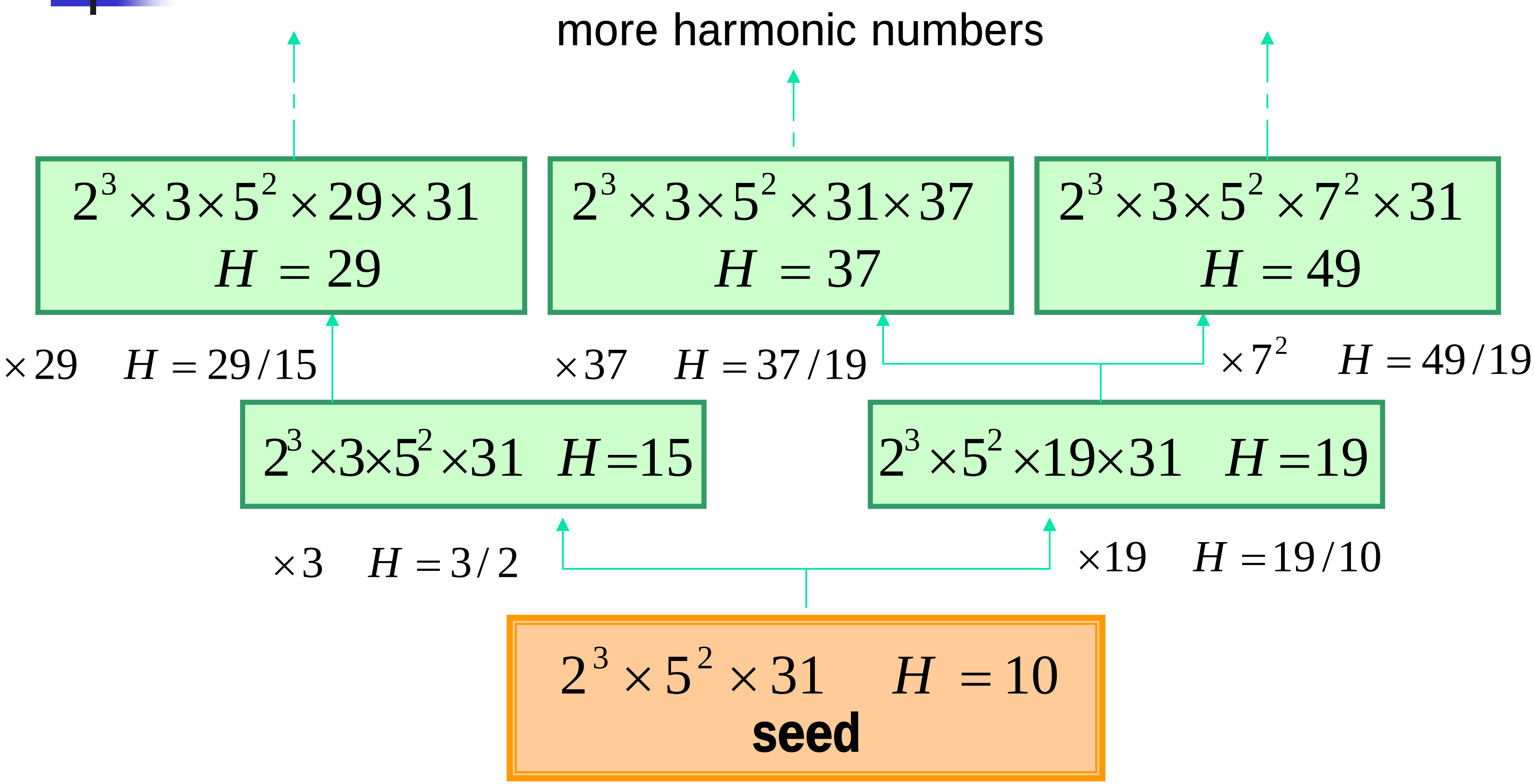
Let $t(n)$=the number of divisors of n.
If $(H(n),t(n))=1$, then $H(n) \mid n$.

↓

In Step 3. we can cut almost all possibilities.

※ We used UBASIC program which is useful to factor integers.

Harmonic Seed (1)



Harmonic Seed (2)

It was conjectured that the harmonic seed of a harmonic number is unique, but we found the following counterexample.

$$2^4 \times 3 \times 5^2 \times 7^2 \times 19 \times 31^2 \times 83 \times 331 \qquad H = 525$$

$\times 5^2 \quad H = 75/31$

$\times 3 \quad H = 3/2$

$$2^4 \times 3 \times 7^2 \times 19 \times 31^2 \times 83 \times 331$$

$H = 217$ **seed**

$$2^4 \times 5^2 \times 7^2 \times 19 \times 31^2 \times 83 \times 331$$

$H = 350$ **seed**