On Ore's harmonic numbers

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Contents

Notation

- n: a positive integer
- The divisor function $\sigma_k(n) := \sum_{d|n} d^k$

  $\sigma_1(n) =$ the sum of divisors of n

  $\sigma_0(n) =$ the number of divisors of n

- $\sigma_k(n)$ is multiplicative, i.e.

  $(n, m) = 1 \Rightarrow \sigma_k(nm) = \sigma_k(n)\sigma_k(m)$

Notation

H(n) := harmonic mean of divisors of n.

Example.

$$H(6) = \frac{4}{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{6}} = \boxed{\frac{4\times 6}{6+3+2+1}} = 2$$

- The function H is also multiplicative since

$$H(n) = \frac{n\,\sigma_0(n)}{\sigma_1(n)}$$

Definition of (Ore's) Harmonic Numbers

A positive integer n is said to be harmonic if H(n) is integral.

- 1 is called a trivial harmonic number.
- 6 is the smallest nontrivial harmonic number.
- (Remark) The following rational number is also called (n-th) harmonic number.

$$H_n := \sum_{i=1}^{n} \frac{1}{i}$$

Ore's Theorem

Ore proved that

every perfect number is harmonic.

Proof. Let n be a perfect number. Then

$$H(n) = \frac{n\,\sigma_0(n)}{\sigma_1(n)} = \frac{\sigma_0(n)}{2}.$$

It is easy to show that this is integral for a perfect number n.

Ore’s Conjecture

Ore’s conjecture:

Every nontrivial harmonic number is even. (??)

↓ if true

There does not exist an odd perfect number!

Table of Harmonic numbers

n	H(n)	n	H(n)	n	H(n)	n	H(n)	n	H(n)
1	1	18600	15	360360	44	2290260	41	15495480	86
6	2	18620	14	539400	44	2457000	60	16166592	51
28	3	27846	17	695520	29	2845800	51	17428320	96
140	5	30240	24	726180	46	4358600	37	18154500	75
270	6	32760	24	753480	39	4713984	48	23088800	70
496	5	55860	21	950976	46	4754880	45	23569920	80
672	8	105664	13	1089270	17	5772200	49	23963940	99
1638	9	167400	19	1421280	42	6051500	50	27027000	110
2970	11	173600	27	1539720	47	8506400	49	29410290	81
6200	10	237510	25	2178540	47	8872200	53	32997888	84
8128	7	242060	29	2178540	54	11981970	77	33550336	13
8190	15	332640	26	2229500	35	14303520	86	…………….	……

Known Facts

Problem

Which values does the harmonic mean take? Are there integers n satisfying $H(n)=4,12,16,18,20,22,\ldots$?

Theorem (Kanold, 1957)

For any positive integer c, there exist only finitely many numbers n satisfying $H(n)=c$.

Known Facts

In 1997, Cohen listed

all harmonic numbers n satisfying H(n) $\leqq$ 13.

H(n)	n	H(n)	n
1	1	8	672
2	6	9	1638
3	28	10	6200
5	140	11	2970
	496	13	105664
6	270		33550336
7	8128		

⇒ There does not exist a harmonic number n with H(n)=4 or 12.

Main Result

An algorithm to find all harmonic numbers n satisfying H(n)=c for a given integer c.

↓ Use of a personal computer

The list of all harmonic numbers n satisfying $H(n) \leqq 1200$ (there are 1376 such numbers).

↓ Particular result

If n is a nontrivial odd harmonic number, then $H(n) > 1200$.

Nonexistence of Odd One

There are no odd perfect numbers less than 10^{300}. (Brent, Cohen & te Riele, 1991)

There are no odd nontrivial harmonic numbers less than 10^{12}. (Cohen & Sorli, 1998)

There are exactly 435 harmonic numbers less than 10^{12}. (Sorli, Ph.D thesis)

Application of the Main Result

(my method)

- List all harmonic numbers n satisfying $H(n) < 1000$.
- List all harmonic numbers n satisfying $H(n)^4 > n$.
- The union of these lists contains all harmonic numbers less than 10^{12}, because $n<10^{12} \Rightarrow H(n)<1000$ or $H(n)^4>n$.

By $1000 \rightarrow 1200$ and $4 \rightarrow 4.25$, we obtain the fact:

there are exactly 633 harmonic numbers less than 10^{13}.

New Record of the nonexistence

There are no nontrivial odd harmonic numbers less than 10^{13}.

There are no nontrivial odd harmonic numbers less than 10^{15}. (Sorli, Ph.D thesis)

Another Application

Ω(n):= the total number of primes dividing n

$$(n = p_1^{e_1} \cdots p_r^{e_r} \;\rightarrow\; \Omega(n) = e_1 + \cdots + e_r)$$

If n is an odd perfect number, then
Ω(n) ≧ 37. (Iannucci and Sorli, 2003)

If n is a nontrivial odd harmonic number,
then **Ω**(n) ≧ 11.

Proof. $\Omega(n) \leq 10 \;\Rightarrow\; \sigma_0(n) \leq 2^{10} = 1024$

$$\Rightarrow\; H(n) < 1024.$$

About Distinct Prime Factors

$\boldsymbol{\omega}(n) :=$ the number of distinct primes dividing n

If n is an odd perfect number, then

$\boldsymbol{\omega}(n) \geqq 8$. (Chein, 1979, Hagis, 1980)

If n is a nontrivial odd harmonic number, then

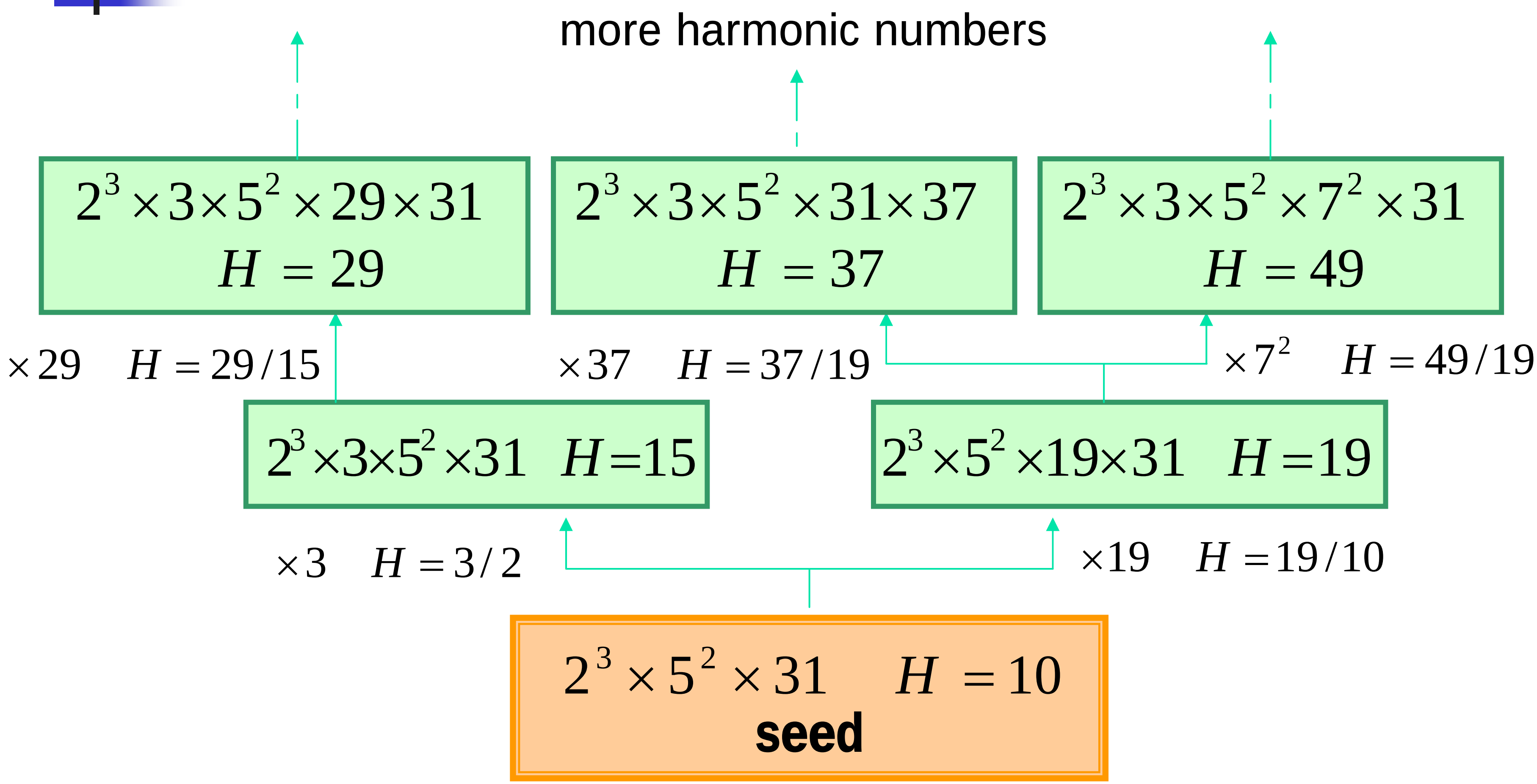
$\boldsymbol{\omega}(n) \geqq 3$. (Pomerance, 1973)

Open Problem

Problem. If $n=p^a q^b r^c$ is harmonic, is it even?

- Maybe, this problem can be solved by using the theory of cyclotomic polynomials.
- In the case of perfect numbers, it is known that there exist only finitely many odd perfect numbers n with a fixed $\omega(n)$ (in fact, $n<2^{(4^{\omega(n)})}$), but I know no analogous fact in the case of harmonic numbers (or seeds).

Harmonic Seeds



Harmonic Seeds

It was conjectured that the harmonic seed of a harmonic number is unique, but we found the following counterexample.

$$2^4 \times 3 \times 5^2 \times 7^2 \times 19 \times 31^2 \times 83 \times 331 \qquad H = 525$$

$\times 5^2 \quad H = 75/31$

$\times 3 \quad H = 3/2$

$2^4 \times 3 \times 7^2 \times 19 \times 31^2 \times 83 \times 331$

$H = 217$ **seed**

$2^4 \times 5^2 \times 7^2 \times 19 \times 31^2 \times 83 \times 331$

$H = 350$ **seed**

Basic Algorithm

Let $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ be the factorization of n.
We call $(e_1, e_2, \cdots, e_k)$ the pairs of exponents of n.

Step 1. List possible pairs of exponents.

For example, if H(n)=5, possible triple pairs are (1,1,1) and (2,1,1).

proof. If n has the pair (3,1,1), then

$$H(n) \geq H(2^3)H(3)H(5) = 16/3 > 5,$$

a contradiction. Similarly, (2,2,1) is an impossible pair.

Basic Algorithm

Step 2. List possible prime factors.

For example, suppose that

- H(n)=5,
- the pair of exponents of n is (2,1,1),
- p is the smallest prime factor of n.

Then $\sigma_{-1}(n) = \dfrac{\sigma_0(n)}{H(n)} = \dfrac{12}{5}.$

On the other hand, $\sigma_{-1}(n) < \left(\frac{p}{p-1}\right)^3.$

Hence $p < (1 - \sqrt[3]{5/12})^{-1} = 3.95.$

Improved Algorithm

Proposition Let c be the numerator of H(n).

$$(c, \sigma_0(n)) = d \ \Rightarrow \ (c/d) \mid n.$$

↓

In Step 2, we can cut many possibilities.

※ We used UBASIC program which is useful to factor integers.