Lemma 2.3.pdf, a short proof of the theorem:

Suppose that \( n \) is an odd harmonic number and \( \omega(n) = k \). Then \( n \leq (2^k)^{2k(k+1)} \). In particular, there exist only finitely many odd harmonic numbers \( n \) satisfying \( \omega(n) = k \), for a given positive integer \( k \).

We define the divisor function \( \sigma_k \) by \( \sigma_k(n) = \sum_{d|n} d^k \).

**Lemma 1 ([1]).** Let \( n \) be a positive odd integer. Suppose that \( \omega(n) = k \) and \( \sigma_{-1}(n) = n/d \) with \( n, d \in \mathbb{N} \). Then \( n < (d+1)^{4k} \).

**Lemma 2.** Let \( n \) be an odd integer greater than 1 with \( \omega(n) = k \), and \( \delta \) a real number less than 1. Then \( \sigma_0(n) < n^\delta / \delta^k \).

**Proof.** Suppose that the canonical factorization of \( n \) is \( \prod_{i=1}^{k} p_i^{e_i} \). Then

\[
\frac{\sigma_0(n)}{n^\delta} = \prod_{i=1}^{k} \frac{e_i + 1}{p_i^{e_i \delta}}.
\]

When \( p \geq 3 \) and \( x > 0 \), we easily see \( p^x > x + 1 \). Hence

\[
\frac{e + 1}{p^\delta} < \frac{e + 1}{e \delta + 1} < \frac{e + 1}{e \delta + \delta} = \frac{1}{\delta}
\]

and therefore,

\[
\frac{\sigma_0(n)}{n^\delta} < \frac{1}{\delta^k},
\]

as required. \( \square \)

**Proof of the Theorem.** If \( n = 1 \), then the required inequation clearly holds. Assume that \( n \) is an odd harmonic number greater than 1. Since

\[
\sigma_{-1}(n) = \frac{\sigma_0(n)}{H(n)} > 1,
\]

Lemmas 1 and 2 imply that

\[
n < (H(n) + 1)^{4k} \leq \sigma_0(n)^4k < \frac{n^{4k-4k}}{\delta^{k-4k}}
\]

for any real number \( \delta < 1 \). Hence

\[
n^{1-4k} < \frac{1}{\delta^{k-4k}},
\]

therefore, if \( \delta < 1/4^k \), then

\[
n < \delta^{k-4k}/(1-\delta^{4k}).
\]

In particular, when we put \( \delta = 1/2^{2k+1} \), we have

\[
n < (2^{2k+1})^{2k-4k} = (2^4)^{2k(k+1)},
\]

as required. \( \square \)

**References**