A mixed joint universality theorem for zeta-functions
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In 1975, S. M. Voronin observed a very interesting property of the Riemann zeta-function \( \zeta(s) \), \( s = \sigma + it \). He proved that every analytic non-vanishing function can be approximated with a given accuracy uniformly on compact subsets of the strip \( D = \{ s \in \mathbb{C} : \frac{1}{2} < \sigma < 1 \} \) by shifts \( \zeta(s + i\tau) \). The last version of the Voronin theorem is the following statement.

**Theorem 1** [2]. Suppose that \( K \) be a compact subset of the strip \( D \) with connected complement, and that \( f(s) \) is a continuous non-vanishing function on \( K \) which is analytic in the interior of \( K \). Then, for every \( \varepsilon > 0 \),

\[
\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0; T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0.
\]

Where \( \text{meas}\{A\} \) denotes the Lebesgue measure of a measurable set \( A \subset \mathbb{R} \).

Later, many authors obtained that some other zeta- and \( L \)-functions also have the above universality property. Among them, the Hurwitz zeta-function \( \zeta(s, \alpha) \) with transcendental parameter \( \alpha \), \( 0 < \alpha \leq 1 \) [3]. However, in this case, the approximated function \( f(s) \) is not necessarily non-vanishing on \( K \).

Y. V. Linnik and I. A. Ibragimov conjectured that all functions given by Dirichlet series and satisfying some natural conditions are universal.

The first result on the joint universality also belongs to S. M. Voronin. In 1975, he proved [7] the joint universality of Dirichlet \( L \)-functions with pairwise non-equivalent characters. In 2007, 2008, H. Mishou obtained [5], [6] the joint universality of the functions \( \zeta(s) \) and \( \zeta(s, \alpha) \). This case is very interesting because the function \( \zeta(s) \) has the Euler product over primes while \( \zeta(s, \alpha) \), in general, has not the Euler product.

In 2006, A. Javtokas and A. Laurinčikas began to study the periodic Hurwitz zeta-function which is a generalization of the function \( \zeta(s, \alpha) \). Let \( \alpha = \{a_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \} \) be a periodic sequence of complex numbers with minimal period \( k \in \mathbb{N} \), and \( \alpha \), \( 0 < \alpha \leq 1 \), be a fixed parameter. Then, the periodic Hurwitz zeta-function \( \zeta(s, \alpha; \alpha) \) is defined, for \( \sigma > 1 \), by

\[
\zeta(s, \alpha; \alpha) = \sum_{m=0}^{\infty} \frac{a_m}{(m + \alpha)^s},
\]

and is meromorphically continued to the whole complex plane. The mentioned authors obtained [1] the universality of the function \( \zeta(s, \alpha; \alpha) \) with transcendental parameter \( \alpha \).

A. Laurinčikas and his students also investigated the joint universality of periodic Hurwitz zeta-functions. The most general result is contained in [4]. For \( j = 1, \ldots, r \), let \( \alpha_j \), \( 0 < \alpha_j \leq 1 \), be a fixed parameter, \( l_j \in \mathbb{N} \), and, for \( j = 1, \ldots, r \), \( l = 1, \ldots, l_j \), let \( \alpha_{jl} = \{a_{mjl} : m \in \mathbb{N}_0 \} \) be a periodic sequence of complex numbers with minimal period \( k_{jl} \in \mathbb{N} \), and \( \zeta(s, \alpha_j; \alpha_{jl}) \) denote the corresponding periodic Hurwitz zeta-function. Denote by \( k_j \) be the least common multiple of the periods \( k_{j1}, \ldots, k_{jl_j} \),
\( j = 1, ..., r \), and define

\[
B_j = \begin{pmatrix}
    a_{1j1} & a_{1j2} & \cdots & a_{1jl_j} \\
    a_{2j1} & a_{2j2} & \cdots & a_{2jl_j} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{kj1} & a_{kj2} & \cdots & a_{kjl_j}
\end{pmatrix}, \quad j = 1, ..., r.
\]

**Theorem 2** [4]. Suppose that the system \{log\((m + \alpha_j) : m \in \mathbb{N}_0, j = 1, ..., r\)\} is linearly independent over the field of rational numbers \(\mathbb{Q}\), and that rank(\(B_j\)) = \(l_j\), \(j = 1, ..., r\). For every \(j = 1, ..., r\) and \(l = 1, ..., l_j\), let \(K_{jl}\) be a compact subset of the strip \(D\) with connected complement, and let \(f_{jl}(s)\) be a continuous on \(K_{jl}\) function which is analytic in the interior of \(K_{jl}\). Then, for every \(\epsilon > 0\),

\[
\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0; T] : \sup_{1 \leq j \leq r} \sup_{1 \leq l \leq l_j} \sup_{s \in K_{jl}} |\zeta(s + i\tau, \alpha_j; A_{jl}) - f_{jl}(s)| < \epsilon \right\} > 0.
\]

In the report, we present a mixed joint universality theorem for the Riemann zeta-function and periodic Hurwitz zeta-functions \(\zeta(s, \alpha_j; A_{jl}), j = 1, ..., r, l = 1, ..., l_j\).

**Theorem 3.** Suppose that \(\alpha_1, ..., \alpha_r\) are algebraically independent over \(\mathbb{Q}\), and that all hypotheses on \(K_{jl}\) and \(f_{jl}\) of Theorem 2 are satisfied. Moreover, let \(K\) be a compact subset of the strip \(D\) with connected complement, and let \(f(s)\) be a continuous non-vanishing on \(K\) function which is analytic in the interior of \(K\). Then, for every \(\epsilon > 0\),

\[
\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0; T] : \sup_{1 \leq j \leq r} \sup_{1 \leq l \leq l_j} \sup_{s \in K_{jl}} |\zeta(s + i\tau, \alpha_j; A_{jl}) - f_{jl}(s)| < \epsilon, \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \epsilon \right\} > 0.
\]

**References**


